**THE WORKS OF ARCHIMEDES**

Translated by Thomas Heath

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THE WORKS OF ARCHIMEDES INCLUDING THE METHOD

BIOGRAPHICAL NOTE  
**ARCH1N1EDES,** c. 287-212 **B.C.**

ARCHIMEDES was a citizen of Syracuse, in Sicily, where he was born around the year 287 **13.C.** IIe was intimate with Hiero, King of Syracuse, and with his son, Gelo, and Plutarch says that he was related to them. In his *Sand-Reckoner,* which was dedicated to Gelo, Archimedes speaks of his father, Pheidias, as an astronomer who investigated the sizes and distances of the sun and moon.

As a young man Archimedes seems to have spent some time in Egypt, where he invented the water-screw as a means of drawing water out of the Nile for irrigating the fields, though it is also said that he invented this machine to drain bilge water from a huge ship built for King Hiero. He may have studied with the pupils of Euclid in Alexandria. It was probably there that he made the friendship of Conon of Samos and Eratosthenes. To Conon he was in the habit of communicating his discoveries before their publication, and it was for Era-tosthenes that he wrote the *Method* and through him that he addressed the famous *Cattle-Problem* to the mathematicians of Alexandria—if the tradition is to be credited that associates Archimedes with this problem. After the death of Conon, Archimedes sent his discoveries to Conon's friend and pupil, Dositheus of Pelusium, to whom four of the extant treatises are dedicated.

His mechanical inventions won great fame for Archimedes and figure largely in the traditions about him. After discovering the solution of the problem *To move a given weight by a given force,* he boasted to King Hiero: "Give me a place to stand on and I can move the earth." Asked for a practical demonstration, he contrived a machine by which with the use of only one arm he drew out of the dock a large ship, laden with passengers and goods, which the combined strength of the Syracusans could scarcely move. From that day Hiero ordered that "Archimedes was to be believed in everything he might say." At the king's re­quest Archimedes then made for him catapults, battering rams, cranes, and many other engines of war, which were later used with such success in the defense of Syracuse against the Romans that they were unable to take the city except by treachery. There is also a story in Lucian that Archimedes set fire to the Roman ships by an arrangement of burning glasses.

Although Archimedes acquired by his mechanical inventions "the renown of more than human sagacity," according to Plutarch, he "would not deign to leave behind him any commentary or writing on such subjects," since he con­sidered them "sordid and ignoble." He did, however, write a description, now lost, of an apparatus, composed of concentric glass spheres moved by water power, representing the Eudoxian system of the world. This astronomical ma­chine, which survived to be seen and described by Cicero in his *Republic,* was sufficiently accurate to show the eclipses of the sun and the moon. Except for this lost work *On Sphere-making,* Archimedes wrote only on strictly mathemat­ical subjects. Ile took all the mathematical sciences for his province: arithme­tic, geometry, astronomy, mechanics, and hydrostatics. 1Tnlike Euclid and Apol-

I310GRAPIIICAL NOTE

lonius he wrote no textbooks. Of his writings, although some have been lost, the most important have survived.

The absorption of Archimedes in his mathematical investigations was so great that he forgot his food and neglected his person, and when carried by force to the bath, Plutarch records, "he used to trace geometrical figures in the ashes of the fire and diagrams in the oil on his body." Asked by Hiero to dis­cover whether a goldsmith had alloyed with silver the gold of his crown, Ar­chimedes found the answer while bathing by considering the water displaced by his body, whereupon he is reported to have run home in his excitement with­out his clothes, shouting, "Eureka" (I have found it).

Archimedes' preoccupation with mathematics is even said to have been the cause of his death. In the general massacre which followed the capture of Syra­cuse by Marcellus in 212 }lc., Archimedes was so intent upon a mathematical diagram that he took no notice, and when ordered by a soldier to attend the victorious general, he refused until he should have solved his problem, where­upon he was slain by the enraged soldier. No blame attaches to the Roman general, Marcellus, since he had given orders to spare the house and person of the mathematician, and in the midst of his triumph he lamented the death of Archimedes, provided him with an honorable burial, and befriended his sur­viving relatives. In accordance with the expressed desire of Archimedes, his family and friends inscribed on his tomb the figure of his favorite theorem, on the sphere and the circumscribed cylinder, and the ratio of the containing solid to the contained. When Cicero was in Sicily as quaestor in 75 **B.C.,** he dis­covered the neglected and forgotten tomb of Archimedes near the Agrigentine Gate and piously restored it.

ON THE SPHERE AND CYLINDER  
BOOK ONE

**ARCHINIEDES to DOSITHEUS** greeting

"On a former occasion I sent you the investigations which I had up to that time completed, including the proofs, showing that any segment bounded by a straight line and a section of a right-angled cone [a parabola] is four-thirds of the triangle which has the same base with the segment and equal height. Since then certain theorems not hitherto demonstrated have occurred to me, and *I* have worked out the proofs of them. They are these: first, that the surface of any sphere is four times its greatest circle; next, that the surface of any seg­ment of a sphere is equal to a circle whose radius is equal to the straight line drawn from the vertex of the segment to the circumference of the circle which is the base of the segment; and, further, that any cylinder having its base equal to-the greatest circle of those in the sphere, and height equal to the diameter of the sphere, is itself *[i.e.* in content] half as large again as the sphere, and its surface also [including its bases] is half as large again as the surface of the sphere. Nov these properties were all along naturally inherent in the figures referred to, but remained unknown to those who were before my time engaged in the study of geometry. Having, however, now discovered that the proper­ties are true of these figures, I cannot feel any hesitation in setting them side by side both with my former investigations and with those of the theorems of Eudoxus on solids which are held to be most irrefragably established, namely, that any pyramid is one third part of the prism which has the same base with the pyramid and equal height, and that any cone is one third part of the cylin­der which has the same base with the cone and equal height. For, though these properties also were naturally inherent in the figures all along, yet they were in fact unknown to all the many able geometers who lived before Eudoxus, and had not been observed by any one. Now, however, it will be open to those who possess the requisite ability to examine these discoveries of mine. They ought to have been published while Conon was still alive, for I should conceive that he would best have been able to grasp them and to pronounce upon them the appropriate verdict; but, as I judge it well to communicate them to those who are conversant with mathematics, I send them to you with the proofs written out, which it will be open to mathematicians to examine. Farewell.

"I first set out the axioms and the assumptions which I have used for the proofs of my propositions."

1

2 ARCHIMEDES

DEFINITIONS

I. "There are in a plane certain terminated bent lines, which either lie wholly on the same side of the straight lines joining their extremities, or have no part of them on the other side."

1. "I apply the term *concave* in *the same direction to* a line such that, if any two points on it are taken, either all the straight lines connecting the points fall on the same side of the line, or some fall on one and the same side while others fall on the line itself, but none on the other side."
2. "Similarly also there are certain terminated surfaces, not themselves be­ing in a plane but having their extremities in a plane, and such that they will either be wholly on the same side of the plane containing their extremities, or have no part of them on the other side."
3. "I apply the term *concave in the same direction* to surfaces such that, if any two points on them are taken, the straight lines connecting the points either all fall on the same side of the surface, or some fall on one and the same side of it while some fall upon it, but none on the other side."
4. "I use the term *solid sector,* when a cone cuts a sphere, and has its apex at the centre of the sphere, to denote the figure comprehended by the surface of the cone and the surface of the sphere included within the cone."
5. "I apply the term *solid rhombus,* when two cones with the same base have their apices on opposite sides of the plane of the base in such a position that their axes lie in a straight line, to denote the solid figure made up of both the cones."

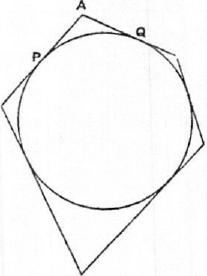
ASSUMPTIONS

1. *"Of all lines which have the same extremities the straight line is the least."*
2. "Of other lines in a plane and having the same extremities, [any two] such are unequal whenever both are concave in the same direction and one of theni is either wholly included between the other and the straight line which has the same extremities with it, or is partly included by, and is partly common with, the other; and that [line] which is included is the lesser [of the two]."
3. "Similarly, of surfaces which have the same extremities, if those extremi­ties are in a plane, the plane is the least [in area]."
4. "Of other surfaces with the same extremities, the extremities being in a plane, [any two] such are unequal whenever both are concave in the same di­rection and one surface is either wholly included between the other and the plane which has the same extremities with it, or is partly included by, and partly common with, the other; and that [surface] which is included is the leSser [of the two in area]."
5. "Further, of unequal lines, unequal surfaces, and unequal solids, the greater exceeds the less by such a magnitude as, when added to itself, can be made to exceed any assigned magnitude among those which are comparable with [it and with] one another.

"These things being premised, *if a polygon be inscribed in a circle, it is ptain that the perimeter of the inscribed polygon is less than the circumference of the circle;* for each of the sides of the polygon is less than that part of the circum­ference of the circle which is cut off by it."

ON THE SPHERE AND CYLINDER I 3

**PROPOSITION I**



*If a polygon be circumscribed about a circle, the perimeter of the circumscribed polygon is greater than the perimeter of the circle.*

Let any two adjacent sides, meeting in *A,* touch the circle at *P, Q* respectively.

Then *[Assumptions,* 2]

*PA-FAQ>* (arc *PQ).*

A similar inequality holds for each angle of the polygon; and, by addition, the required result fol­lows.

**PROPOSITION** 2

*Given two unequal magnitudes, it is possible to find two unequal straight lines such that the greater straight line has to the less a ratio less than the greater magnitude has to the less.*

Let *AB,D* represent the two unequal magnitudes, *AB* being the greater.

**A** Suppose *BC* measured along *BA* equal to *D,* and let *GH* be any straight line.

Then, if *CA* be added to itself a sufficient number of times, the sum will exceed D. Let *AF* be this sum, and take *E* on *GH* produced such that *GH* is the same multiple of *HE* o that *Al?* is of *AC.*

Thus EH *: HG = AC : AF.*But, since *AF>D* (or *CB),*

*AC : AR <.AC :CB.*Therefore, *componendo,*

*El; :GH <.4B : D.*

Hence *EG, GH* are two lines satisfying the given condition.

**PROPOSITION** 3

*Given two unequal magnitudes and a circle, it is possible to inscribe* a *polygon in the circle and to describe another about it so that the side of the circumscribed poly­gon may have to the side of the inscribed polygon a ratio less than that of the greater magnitude to the less.*

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Let A, *B* represent the given magnitudes, *A* being the greater.

Find [Prop. 2] two straight lines *F, KL,* of which *F* is the greater, such that

*: KL* <A :*13. (1).*

4 ARCHIMEDES

Draw *LM* perpendicular to *LK* and of such length that *KM = F.*

In the given circle let *CE, DG* be two diameters at right angles. Then, bisect­ing the angle *DOC,* bisecting the half again, and so on, we shall arrive ulti­mately at an angle (as *NOC)* less than twice the angle *LKM.*

Join *NC,* which (by the construction) will be the side of a regular polygon inscribed in the circle. Let *OP* be the radius of the circle bisecting the angle *NOC* (and therefore bisecting *NC* at right angles, in *H, say),* and let the tan­gent at *P* meet *OC,* ON produced in *S, T* respectively.

Now, since *L CON <* 2 L *LKM,*

*L HOC < L LKM,*

and the angles at *H, L* are right;

therefore *MK : LK > OC : OH*

*>OP :OH.*

Hence *ST : CY <MK : LK*

*<F : LK;*

therefore, *a fortiori, by (1),*

*ST :CN <A : B.*

*Thus* two polygons are found satisfying the given condition.

**PROPOSITION** 4

*Again, given two unequal magnitudes and a sector, it is possible to describe a poly­gon about the sector and to inscribe another in it so that the side of the circumscribed polygon may have to the side of the inscribed polygon a ratio less than the greater magnitude has to the less.*

[The "inscribed polygon" found in this proposition is one which has for two sides the two radii bounding the sector, while the remaining sides (the number of which is, by construction, some power of 2) subtend equal parts of the arc of the sector; the "circumscribed polygon" is formed by the tangents parallel to the sides of the inscribed polygon and by the two bounding radii produced.]

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| **A** | F | B |
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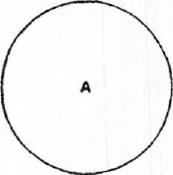
In this case we make the same construction **as in the last proposition** except that we bisect the angle *COD* of the sector, instead of the right angle between two diameters, then bisect the half again, and so on. The proof is exactly similar to the preceding one.

**PROPOSITION** 5

*Given a circle and two unequal magnitudes, to describe a polygon about the circle and inscribe another in it, so that the circumscribed polygon may hare to the in­scribed a ratio less than the greater magnitude has to the less.*

ON THE SPHERE AND CYLINDER I 5

Let ***A*** be the given circle and ***B, C*** the given magnitudes, ***B*** being the greater.



Take two unequal straight lines ***D, E,*** of which ***D*** is the greater,

B c such that ***D : E <B :C*** [Prop. 2],  
and let ***F*** be a mean proportional between ***D, E*** so that ***D*** is also greater than ***F.***

Describe (in the manner of Prop.

E 3) one polygon about the circle,

and inscribe another in it, so that

F the side of the former has to the side  
of the latter a ratio less than the ratio ***D : F.***

Thus the duplicate ratio of the side of the former polygon to the side of the latter is less than the ratio D2 : le2.

But the said duplicate ratio of the sides is equal to the ratio of the areas of the polygons, since they are similar;

therefore the area of the circumscribed polygon has to the area of the in­scribed polygon a ratio less than the ratio D2 : F2, or ***D : E,*** and *a fortiori* less than the ratio ***B : C.***

**PROPOSITION** 6

"Similarly we can show that, *given two unequal magnitudes and a sector, it is possible to circumscribe a polygon about the sector and inscribe in it another similar one* so *that the circumscribed* may *have to the inscribed a ratio less than the greater magnitude has to the less.*

"And it is likewise clear that, ***if*** *a circle or a sector, as well as a certain area, be given, it is possible, by inscribing regular polygons in the circle or sector, and by continually inscribing such in the remaining segments, to leave segments of the circle or sector which are [together] less than the given area.* For this is proved in the *Elements* [Eucl. xii. 2].

"But it is yet to be proved that, *given a circle or sector and an area, it is possible to describe a polygon about the circle or sector, such that the area re­maining between, the circumference and the circumscribed figure is less than the given area."*

The proof for the circle (which, as Archimedes ***says,*** can be equally ap­plied to a sector) is as follows.

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| B |
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| so that |  |

Let *A* be the given circle and ***B*** the given area.

Now, there being two unequal mag­nitudes *A +B* and A, let a polygon *(C)* be circumscribed about the circle and a polygon ***(I)*** inscribed in it [as in Prop. 5],

***C:I<A+B*** *:A.*

The circumscribed polygon *(C)* shall be that required.

**6** ARCHIMEDES

For the circle (A) is greater than the inscribed polygon *(I).*

Therefore, from **(1),** a *fortiori,*

*C :A<A+B* :A,

whence *C<A+B,*

or *C — A <B.*

*PROPOSITION 7*

*If in an isosceles cone* [i.e. *a right circular cone] a pyramid be inscribed having an equilateral base, the surface of the pyramid excluding the base is equal to* a *triangle having its base equal to the perimeter of the base of the pyramid and its height equal to the perpendicular drawn from the apex on one side of the base.*

Since the sides of the base of the pyramid are equal, it follows that the per­pendiculars from the apex to all the sides of the base are equal; and the proof of the proposition is obvious.

**PROPOSITION** 8

*If a pyramid be circumscribed about an isosceles cone, the surface of the pyramid excluding its base is equal to a triangle having its base equal to the perimeter of the base of the pyramid and its height equal to the side* [i.e. *a generator] of the cone.*

The base of the pyramid is a polygon circumscribed about the circular base of the cone, and the line joining the apex of the cone or pyramid to the point of contact of any side of the polygon is perpendicular to that side. Also all these perpendiculars, being generators of the cone, are equal; whence the prop­osition follows immediately.

**PROPOSITION** 9

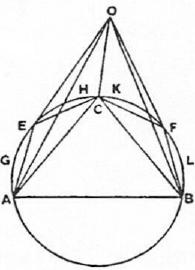
*If in the circular base of an isosceles cone a chord be placed, and from its extremi­ties straight lines be drawn to the apex of the cone, the triangle so formed will be less than the portion of the surface of the cone intercepted between the lines drawn to the apex.*

Let *ABC* be the circular base of the cone, and *0* its apex.

Draw a chord *AB* in the circle, and join *OA, OB.* Bisect the *arc ACB* in *C,* and join *AC, BC, OC.*

Then *AOAC+ AOBC> AOAB.*

Let the excess of the sum of the first two triangles over the third be equal to the area *D.*



Then *D* is either less than the sum of the segments *AEC, CFB,* or not less.

I. Let *D* be not less than the sum of the segments referred to.

We have now two surfaces

1. that consisting of the portion *OAEC* of the surface of the cone together with the segment *AEC,* and
2. the triangle *OAC;*

ON THE SPHERE AND CYLINDER I

and, since the two surfaces have the same extremities (the perimeter of the triangle *OAC),* **the** former surface is greater than the latter, which is *included* by it *[Assumptions,* 3 or 4].

**Hence ,** (surface *OA EC) +* (segment *AEC) > QOAC.*

Similarly (surface *OCFB)+* (segment *CFB) > AOBC.*

**Therefore, since *D is*** not less **than the sum of the segments, we have,** by

addition,,

(surface *OA ECF B) +D > AOAC+ AOBC*

*> QOAB+D,* by hypothesis.

Taking away the common part *D,* we have the required result.

**II.** Let *D* be less than the sum of the segments *AEC, CFB.*

If now we bisect the arcs *AC, CB,* then bisect the halves, and so on, we shall

ultimately leave segments which are together less than *D.* **[Prop. 6]**

**Let** *AGE, EHC, CKF, FLB* be those segments, and join *OE, OF.*

Then, as before,

(surface *OAGE)-1-* (segment *AGE) > AOAE*

and (surface *OEHC)+* (segment *EHC) > AOEC.*

Therefore (surface *OAGHC)+* (segments *AGE, EHC)*

*> AOAE+AOEC*

*> AOAC, a fortiori.*

Similarly for the part of the surface of the cone bounded by *OC, OB* and

the arc *CFB.*

Hence, by addition,

(surface *OAGEHCKFLB)+* (segments *AGE, EHC, CKF, FLB)*

*> AOAC+6,0BC*

*> AO AB +D,* by hypothesis.

But the sum of the segments is less than *D,* and the required result follows.

**PROPOSITION 10**

*If in the plane of the circular base of an isosceles cone two tangents be drawn to the circle meeting in a point, and the points of contact and the point of concourse of the tangents be respectively joined to the apex of the cone, the sum of the two triangles formed by the joining lines and the two tangents are together greater than the in­cluded portion of the surface of the cone.*

**Let** *ABC* be the circular base of the cone, 0 its apex, *AD, BD* the two tan­gents to the circle meeting in *D.* Join *OA, OB, OD.*

Let *ECF* be drawn touching the circle at *C,* the middle point of the arc *ACB,* and therefore parallel to *AB.* **Join** *0.e, QF.*

Then *ED+DF>EF,*

and, adding *AE+FB* to each side,

*AD+DB>AE+EF+FB.*

**Now** *OA, OC, OB,* being generators of the cone, are equal, and they are

respectively perpendicular to the tangents at A, *C, B.*

It follows that

*AOAD+AODB> AOAE+AOEF+ AOFB.*

Let the area *G* be equal to the excess of the first sum over the second.

G **is then either lees, or** not less, than the sum of the spaces *EAHC, FCKB*

**re paining between the** circle and the tangents, which sum we will call *L.*

**I. Let** *G* be not less than *L.*

8 ARCHIMEDES

We have now two surfaces

1. that of the pyramid with apex 0 and base *AEFB,* excluding the face *OAB,*
2. that consisting of the part *OACB* of the surface of the cone together with the segment *ACB.*

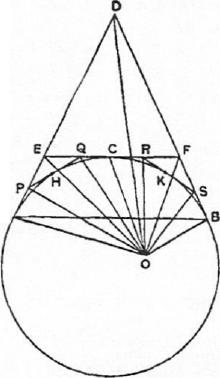
These two surfaces have the same extremities, viz. the perimeter of the tri­angle *OAB,* and, since the former *includes* the latter, the former is the greater *[Assumptions,* 4].

That is, the surface of the pyramid exclusive of the face *OAB* is greater than the sum of the surface *OACB* and the segment *ACB.*

Taking away the segment from each sum, we have

*6,0AE-F-AOEF+ AOFB±L>* the surface *OAIICKB.*

And *G* is not less than *L.*



It follows. that

*L\OAE+ AOEF+ AOFB+G,*

which is by hypothesis equal to *AOAD+ LODB,* is greater than the same surface.

II. Let G be less than *L.*

If we bisect the arcs *AC, CB* and draw tangents at their middle points, then bisect the halves and draw tan­gents, and so on, we shall lastly arrive **A** at a polygon such that the sum of the parts remaining between the sides of the polygon and the circumference of the segment is less than G.

Let the remainders be those be­tween the segment and the polygon *APQRSB,* and let their sum be *M.* Join *OP, OQ,* etc.

Then, as before,

*AOAE+ AOEF-1- AOFB> AOAP+ AOPQ-1- • • -• AOSB.*

Also, *as* before,

(surface of pyramid *OAPQRSB* excluding the face *OAB) >* the part *OACB* of the surface of the cone together with the segment *ACB.*

Taking away the segment from each sum,

*AOAP+ AOPQ-1- • • • >* the part *OACB* of the surface of the cone.

Hence, *a fortiori,*

*AOAE-1-AOEF-i-A0F13-1-G,*

which is by hypothesis equal to

*LOAD -{- AODB,*

is greater than the part *OACB* of the surface of the cone.

**PROPOSITION 11**

*If* a *plane parallel to the axis of a right cylinder cut the cylinder, the part of the surface of the cylinder cut off by the plane is greater than the area of the parallel­ogram in which the plane cuts it.*

ON THE SPHERE AND CYLINDER **I 9**

PROPOSITION 12

*If at the extremities of two generators of any right cylinder tangents be drawn to the circular bases in the planes of those bases respectively, and if the pairs of tangents meet, the parallelograms formed by each generator and the two corresponding tan­gents respectively are together greater than the included portion of the surface of the cylinder between. the two generators.*

[The proofs of these two propositions follow exactly the methods of Props. 9, 10 respectively, and it is therefore unnecessary to reproduce them.]

"From the properties thus proved it is clear (1) that, *if a pyramid be in­scribed in an isosceles cone, the surface of the pyramid excluding the base is less than the surface of the cone [excluding the base],* and (2) that, *if a pyramid be circumscribed about an isosceles cone, the surface of the pyramid excluding the base is greater than the surface of the cone excluding the base.*

"It is also clear from what has been proved both (1) that, *if a prism be in­scribed in a right cylinder, the surface of the prism made up of its parallelograms* [i.e. *excluding its bases] is less than the surface of the cylinder excluding its bases,* and (2) that, if *a prism be circumscribed about a right cylinder, the surface of the prism made up of its parallelograms is greater than the surface of the cylinder excluding its bases."*

PROPOSITION 13

*The surface of any right cylinder excluding the bases is equal to a circle whose radius is a mean proportional between the side* [i.e. *a generator] of the cylinder and the diameter of its base.*

Let the base of the cylinder be the circle A, and make *CD* equal to the diam­eter of this circle, and *ER* equal to the height of the cylinder.

Let *H* be a mean proportional between *CD, EF,* and *B* a circle with radius equal to *H.*

M

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Then the circle *B* shall be equal to the surface of the cylin­der (excluding the bases), which we will call *S.*

For, if not, *B* must be either greater or less than *S.*

**I.** Suppose *B <S.*

Then it is possible to circum­scribe a regular polygon about *B,* and to inscribe another in it, such that the ratio of the former to the latter is less than the ratio *S : B.*

Suppose this done, and cir­cumscribe about *A a* polygon similar to that described about *B;* then erect on the polygon about A a prism of the same height as the cylinder. The prism will therefore be circumscribed to the cylinder.

Let *KD,* perpendicular to *CD,* and *FL,* perpendicular to *EF,* be each equal to the perimeter of the polygon about *A.* Bisect *CD* in *M,* and join *MK.*

10 ARCHIMEDES

Then ***AKDM = the*** polygon about ***A.***

Also ***OEL=*** surface of prism (excluding bases).

Produce ***FE*** to ***N*** so that ***FE= EN,*** and join ***NL.***

Now the polygons about ***A, B,*** being similar, are in the duplicate ratio .of

the radii of ***A, B.***

Thus

***PKDM :*** (polygon about ***B) = MD2 : H2***

***=MD2 :CD • EF***

***=MD :NF***

***= AKDM : ALEN***

(since ***DK =FL).***

Therefore (polygon about ***B) = LLFN***

***=OEL***

= (surface of prism about A),

from above.

But (polygon about ***B) :*** (polygon in ***B) <S : B.***

Therefore

(surface of prism about A) : (polygon in ***B) <S : B,***

and, alternately,

(surface of prism about A) : *S* < (polygon in ***B) : B;***

which is impossible, since the surface of the prism is greater than 8, while the

polygon inscribed in ***B*** is less than ***B.***

Therefore ***B <8.***

II. Suppose ***B> S.***

Let ***a*** regular polygon be circumscribed about ***B*** and another inscribed in it

so that

(polygon about ***B) :*** (polygon in ***B) <B : S.***

Inscribe in A a polygon similar to that inscribed in ***B,*** and erect a prism on

the polygon inscribed in ***A*** of the same height as the cylinder.

Again, let ***DK, FL,*** drawn as before, be each equal to the perimeter of the

polygon inscribed in ***A.***

Then, in this case,

***AKDM>*** (polygon inscribed in A)

(since the perpendicular from the centre on a side of the polygon is less than

the radius of ***A).***

Also ***LLFN =DEL=*** surface of prism (excluding bases).

Now

(polygon in ***A) :*** (polygon in ***B)= MD' : H2,***

***= AKDM :****LLFN,* as before.

And *AKD.111>* (polygon in A).

Therefore

***LLFN,*** or (surface of prism) > (polygon in ***B).***

But this is impossible, because

(polygon about ***B) :*** (polygon in ***B) <B : S,***

< (polygon about ***B) : S, a fortiori,***

so that (polygon in ***B)> S,***

> (surface of prism), a ***fortiori.***

Hence ***B*** is neither greater nor less than 8, and therefore

***B = S.***

ON THE SPHERE AND CYLINDER I 11

l'itorosrrios 14

*The surface of any isosceles cone excluding the base is equal to a circle whose radius is a mean proportional between the side of the cone* [a generator] *and the radius of the circle which is the base of the cone.*

Let the circle A be the base of the cone; draw *C* equal to the radius of the circle, and ***D*** equal to the side of the cone, and let *E* be a mean proportional between *C,* ***D.***

Draw a circle ***B*** with radius equal to *E.*



Then shall ***B*** be equal to the surface of the cone (excluding the base), which we will call *S.*

If not, ***B*** must be either greater or less than *S.* I. Suppose *B <S.*

Let a regular polygon be described about ***B*** and a similar one inscribed in it such that the former has to the latter a ratio less than the ratio *S :* ***B.*** Describe about *A* another similar polygon, and on it set up a pyramid with apex the same as that of the cone.

Then (polygon about A) : (polygon about ***B)***

= C2 : E2  
= *C* : *D*

= (polygon about A) : (surface of pyramid excluding base).

Therefore

(surface of pyramid = (polygon about *B).*

Now (polygon about *B) :* (polygon in *B) <S B.*

Therefore

(surface of pyramid) : (polygon in *B) <S : B,*

which is impossible, (because the surface of the pyramid is greater than *S,*

while the polygon in *B* is less than *B).*

Hence *B <S.*

II. Suppose *B> S.*

Take regular polygons circumscribed and inscribed to *B* such that the ratio

of the former to the latter is less than the ratio *B : S.*

Inscribe in A a similar polygon to that inscribed in *B,* and erect a pyramid

on the polygon inscribed in *A* with apex the same as that of the cone.

In this case

(polygon in *A) :* (polygon in *B) = C2 E2*

*=C : D*

> (polygon in A) : (surface of pyramid excluding base).

This is clear because the ratio of *C* to *D* is greater than the ratio of the

perpendicular from the centre of *A* on a side of the polygon to the perpen-

dicular from the apex of the cone on the same side.

Therefore

(surface of pyramid) > (polygon in *B).*

But (polygon about *B) :* (polygon in *B)* ***<B :*** *S.*

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Therefore, a *fortiori,*

(polygon about *B) :* (surface of pyramid) *<B : S,*

which is impossible.

Since therefore *B* is neither greater nor less than *S,*

*B = S.*

**PROPOSITION** 15

*The surface of any isosceles cone has the same ratio to its base as the side of the cone has to the radius of the base.*

*By* Prop. 14, the surface of the cone is equal to a circle whose radius is a mean proportional between the side of the cone and the radius of the base.

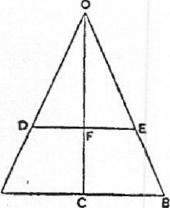
Hence, since circles are to one another as the squares of their radii, the prop­osition follows.

**PROPOSITION** 16

*If an isosceles cone be cut by a plane parallel to the base, the portion of the surface of the cone between the parallel planes is equal to a circle whose radius is* a *mean proportional between (1) the portion of the side of the cone intercepted by the paral­lel planes and* (2) *the line which is equal to the sum of the radii of the circles in the parallel planes.*

Let *OAR* be a triangle through the axis of a cone, *DE* its intersection with the plane cutting off the frustum, and *OFC* the axis of the cone.

Then the surface of the cone *OAB* is equal to a circle



whose radius is equal to *VOA* • *AC.* [Prop. 14.]

Similarly the surface of the cone *ODE* is equal to a

circle whose radius is equal to *VOD•DF.*

And the surface of the frustum is equal to the differ-

ence between the two circles.

Now

*OA •AC —OD -DF =DA -AC--1-0D •AC —OD -DR.*

But *OD• AC =OA • DP,*

since *OA : AC =OD : DF.* **A**

Hence *OA • AC —0D• DR =DA-AC-I-DA • DF*

*= DA • (AC+DF).*

And, since circles are to one another as the squares of their radii, it follows that the difference between the circles whose radii are *VOA AC, \/OD • DF* respec­tively is equal to a circle whose radius is *VDA* • (A *C-FDF).*

Therefore the surface of the frustum is equal to this circle.

LEMMAS

"1. *Cones having equal height have the same ratio as their bases; and those having equal bases have the same ratio as their heights'.*

2. *If a cylinder be cut by a plane parallel to the base, then, as the cylinder is to the cylinder, so is the axis to the axis2.*

'Euclid **XII. 11.** "Cones and cylinders of equal height are to one another as their bases." Euclid xit. 14. "Cones and cylinders on equal bases are to one another as their heights."

=Euclid **XII.** 13. "If a cylinder be cut by a plane parallel to the opposite planes (the bases], then, as the cylinder is to the cylinder, so will the axis be to the axis."

ON THE SPHERE AND CYLINDER I 13

1. *The cones which have the same bases as the cylinders [and equal height] are in the same ratio as the cylinders.*
2. *Also the bases of equal cones are reciprocally proportional to their heights; and those cones whose bases are reciprocally proportional to their heights are equal.'*
3. *Also the cones, the diameters of whose bases have the same ratio as their axes,*

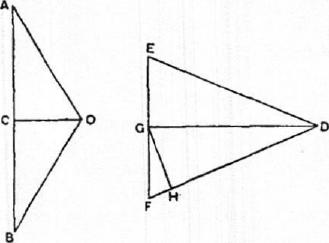
*are to one another in the triplicate ratio of the diameters of the bases.2*

And all these propositions have been proved by earlier geometers."

**PROPOSITION** *17*

*If there be two isosceles* cones, *and the surface of one cone be equal to the base of the other, while the perpendicular from the centre of the base [of the first cone] on the side of that cone is equal to the height [of the second], the cones will be equal.*

Let *OA B, DEF* he triangles through the axes of two cones respectively, *C, G* the centres of the respective bases, *GH* the perpendicular from *G* on *FD;* and suppose that the base of the cone *0 AB is* equal to the surface of the cone *DEF,* and that *OC = OH.*



Then, since the base of *OAB is* equal to the surface of *DEF,*

(base of cone *OAR) :* (base of cone *DEF)*

= (surface of *DEF) :* (base of *DEF)*

*= DF : FG* [Prop. 15]  
*= DO :Gil,* by similar triangles, *=DG :0C.*

Therefore the bases of the cones are reciprocally proportional to their heights; whence the cones are equal. *[Lemma* 4.]

**PROPOSITION 18**

Any *solid rhombus consisting of isosceles cones is equal to the cone which has its base equal to the surface of one of the cones composing the rhombus and its height equal to the perpendicular drawn from the apex of the second cone to one side of*

*the first cone.*

Let the rhombus be *OABD* consisting of two cones with apices ***0, D*** and

with a common base (the circle about *AB* as diameter).

Let, *FHK* be another cone with base equal to the surface of the cone *OAB*

and height *FG* equal to *DE,* the perpendicular from D on *OB.*

Then shall the cone *FHK* be equal to the rhombus.

Construct a third cone *LMN* with base (the circle about *MN)* equal to the

base of *OAB* and height *LP* equal to *OD.*

'Euclid xiI. 15. "The bases of equal cones and cylinders are reciprocally proportional to their heights; and those cones and cylinders whose bases are reciprocally proportional to their heights are equal."

\*Euclid xtt. 12. "Similar cones and cylinders are to one another in the triplicate ratio of the diameters of their bases."

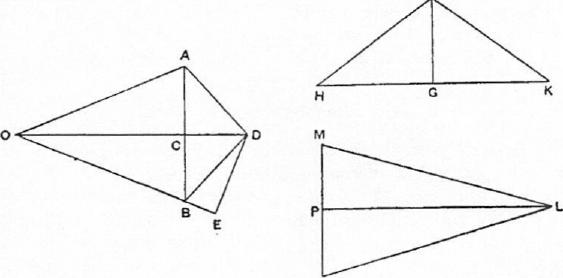
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Then, since *LP* = *OD,*

*LP* : *CD=OD :CD.*

But *[Lemma 1] OD : CD =* (rhombus *OADB) :* (cone *DAB),*

and *LP* : *CD=* (cone *LMN) :* (cone *DAB).* It follows that



N

(rhombus *0.4 DB) =* (cone *LMN).* **(1)**

Again, since *A B = MN,* and

(surface of *OAR) =* (base of *I? HK),*

(base of *PIIK)* : (base of *LAIN) =* (surface of *OA B)* : (base of *OAR)*

*=0B : BC* [Prop. 15]

*=OD : DE,* by similar triangles,

*= LP : FG,* by hypothesis.

Thus, in the cones *FHK, LMN,* the bases are reciprocally proportional to

the heights.

Therefore the cones *FHK, LMN* are equal,

and hence, by (1) the cone *FHK* is equal to the given solid rhombus.

PROPOSITION 19

*If* an *isosceles cone be cut by a plane parallel to the base, and on the resulting circular section a cone be described having as its apex the centre of the base [of the first cone], and if the rhombus so formed be taken away from the whole cone, the part remaining will be equal to the cone with base equal to the surface of the portion of the first cone between the parallel planes and with height equal to the perpen­dicular drawn from the centre of the base of the first cone on one side of that cone.*

Let the cone *OAB* be cut by a plane parallel to the base in the circle on *DE* as diameter. Let *C* be the centre of the base of the cone, and with *C* as apex and the circle about *DE* as base describe a cone, making with the cone *ODE* the rhombus *ODCE.*

Take a cone *FGH* with base equal to the surface of the frustum *DARE* and height equal to the perpendicular *(CK)* from *C* on *AO.*

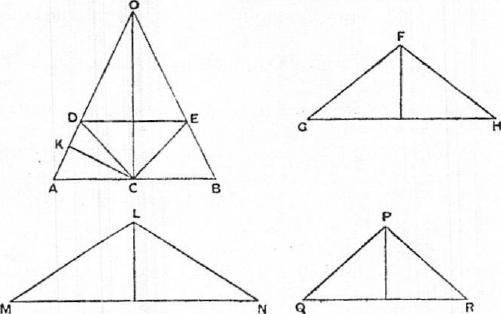
Then shall the cone *FGH* be equal to the difference between the cone *OAB* and the rhombus *ODCE.*

Take (1) a cone *LAIN* with base equal to the surface of the cone *OAR,* and height equal to *CK,*

(2) a cone *PQR* with base equal to the surface of the cone *ODE* and height equal to *CK.*

ON THE SPHERE AND CYLINDER I 15

Now, since the surface of the cone *OAB is* equal to the surface of the cone *ODE* together with that of the frustum *DARE,* we have, by the construction,



(base of *LMN) =* (base of *FGH)+* (base of *PQR)*

and, since the heights of the three cones are equal,

(cone *LMN) =* (cone *FGH)+* (cone *PQR).*

But the cone *LMN* is equal to the cone *OAB* [Prop. 17], and the cone *PQI?*

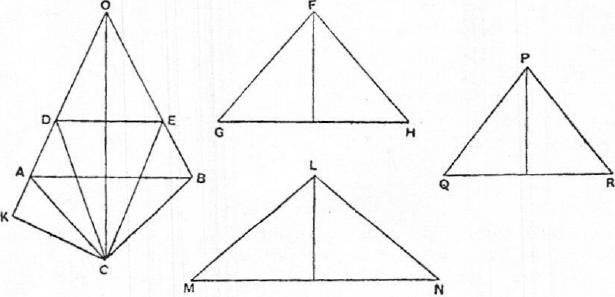
is equal to the rhombus *ODCE* [Prop. 18].

Therefore (cone *OAB) =* (cone *FGH)+* (rhombus *ODCE),* and the proposi-

tion is proved.

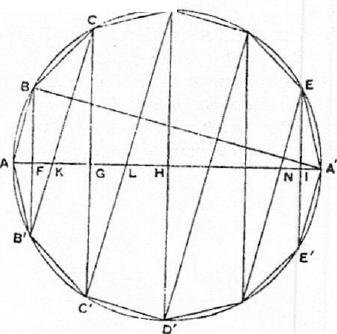
PROPOSITION 20

*If one of the two isosceles cones forming a rhombus be cut by a plane parallel to the base and on the resulting circular section a cone be described having the same apex as the second cone, and if the resulting rhombus be taken from the whole rhombus, the remainder will be equal to the cone with base equal to the surface of the portion of the cone between the parallel planes and with height equal to the perpendicular drawn from the apex of the second cone to the side of the first cone.*



Let the rhombus be *OACB,* and let the cone *OAB* be cut by a plane parallel to its base in the circle about *DE* as diameter. With this circle as base and *C*

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as apex describe a cone, which therefore with *ODE* forms the rhombus *ODCE.*

Take a cone *FGH* with base equal to the surface of the frustum *DABE* and

height equal to the perpendicular *(CK)* from *C* on *OA.*

The cone *FGH* shall be equal to the difference between the rhombi *OACB,*

*ODCE.*

For take (1) a cone *LMN* with base equal to the surface of *OAB* and height

equal to *CK,*

(2) a cone *PQR,* with base equal to the surface of *ODE,* and height equal

t o *CK.*

Then, since the surface of *DAB* is equal to the surface of *ODE* together with

that of the frustum *DABE,* we have, by construction,

(base of *LMN)=* (base of *PQM+* (base of *FGH),*

and the three cones are of equal height;

therefore (cone *LMN) =* (cone *PQR) +* (cone *FGH).*

But the cone *LMN* is equal to the rhombus *OACB,* and the cone *PQR* is

equal to the rhombus *ODCE* [Prop. 18].

Hence the cone *FGH* is equal to the difference between the two rhombi

*OACB, ODCE.*

PROPOSITION 21

*A regular polygon of an even number of sides being inscribed in a circle, as ABC • • •A' • • •C'B' A, so that AA' is a diameter, if two angular points next but one to each other, as B, B', be joined, and the other lines parallel to BB' and joining pairs of angular points be drawn, as CC', DD' • • then (BE-FCC'd- • • .) : AA' =A'B :BA.*

Let *BB', CC', DD', • • •* meet *AA'*

in *F, G, H, • • • ;* and let *CB', DC',*

* *• •* be joined meeting *AA'* in *K, L,*
* *• •* respectively.

Then clearly *CB', DC', • • •* are

parallel to one another and to *AB.*

Hence, by similar triangles,

*BF : FA =B'F : FK*

*=CG :GK*

*=C'G:GL*

*=E'l :IA',*

and, summing the antecedents and consequents respectively, we have *(BB' -{-CC'+ • • .) : AA' =BF : FA*

*=A'B : BA.*

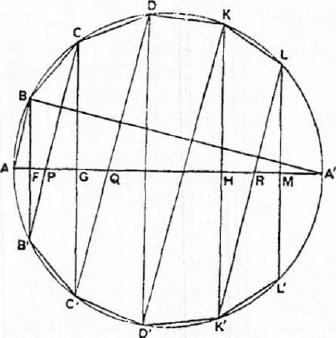
PROPOSITION 22

*If a polygon be inscribed in a segment of a circle LAL' so that all its sides exclud­ing the base are equal and their number even, as LK • • •A • • •K'L', A being the middle point of the segment, and if the lines BB', CC', • • • parallel to the base LL' and joining pairs of angular points be drawn, then*

*(BB'-FCC'-}- • • •-ELM) : AM = A'B : BA,*

*where M is the middle point of LL' and AA' is the diameter through M.*

ON THE SPHERE AND CYLINDER I 17



Joining *CB', DC', • • • LK',* as in the last proposition, and supposing that they meet *AM* in *P, Q, • • • R,* while *BB', CC', • • KK'* meet *AM* in *F, G, • • • II,* we have, by similar tri­angles,

*BF : FA= B'F :FP*

*=CG :PG*

*=C'G :GQ*

*=LM :RM;*

and, summing the antecedents and

consequents, we obtain

*(BB'-FCC'-i- •••A-LM) : AM*

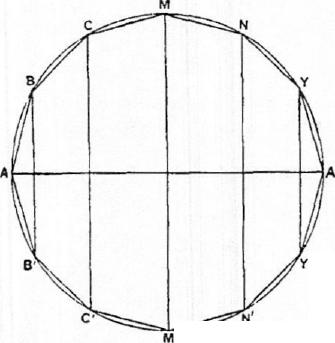
*=BF :FA*

*=A'B :BA.*

PROPOSITION 23

Take a great circle *ABC • • •* of a sphere, and inscribe in it a regular polygon whose sides are a multiple of four in number. Let *AA', MM'* be diameters at right angles and joining opposite angular points of the polygon.

Then, if the polygon and great circle revolve together about the diameter *AA',* the angular points of the polygon, except *A, A',* will de­scribe circles on the surface of the sphere at right angles to the diame­ter *AA'.* Also the sides of the poly­gon will describe portions of conical surfaces, e.g. *BC* will describe a surface forming part of a cone whose base is a circle about *CC'* as diameter and whose apex is the point in which *CB, C'B'* produced meet each other and the diameter *AA'.*



Comparing the hemisphere *MAM'*

* and that half of the figure described by the revolution of the polygon which is included in the hemisphere, we see that the surface of the hemisphere and the surface of the inscribed figure have the same boundaries in one plane (viz. the circle on MM' as diameter), the former surface entirely includes the latter, and they are both concave in the same direction.

Therefore [A*ssumptions,* 41 the surface of the hemisphere is greater than that of the inscribed figure; and the same is true of the other halves of the figures.

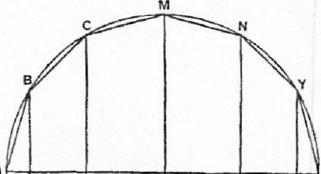
Hence *the surface of the sphere is greater than the surface described by the rev­olution of the polygon. inscribed in the great circle about the diameter of the great circle.*

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**PROPOSITION** 24

***If* a** *regular polygon AB • • •A' • • •B'A, the number of whose sides is a multiple of four, be inscribed in a great circle of a sphere, and if BB' subtending two sides be joined,* and *all the other lines parallel to BB' and joining pairs of angular points be drawn, then the surface of the figure inscribed in the sphere by the revolution of the polygon about the diameter AA' is equal to a circle the square of whose radius is equal to the rectangle*

*BA(BB'+CC'--1- • • .)•*

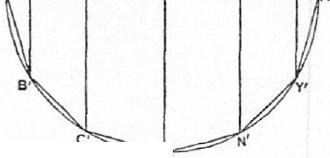


The surface of the figure is made **up** of the surfaces of parts of different cones.

Now the surface of the cone *ABB'*

is equal to a circle whose radius is

*VBA•iBB'.* [Prop. 14] The surface of the frustum *BB'C'C* is equal to a circle of radius



*V BC •t(BB' •-i-CC'),* [Prop. 16] and so on.

It follows, since *BA= BC = • •* that the whole surface is equal to a

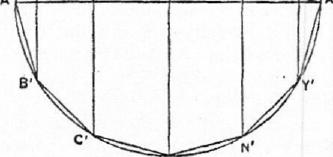
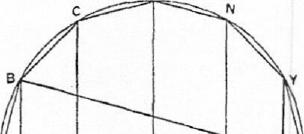
circle whose radius is equal to **N/BA(BBH-CC+ . • •i-M/W-f- • • •-i-Yr).**

**PROPOSITION** 25

*The surface of the figure inscribed in a sphere as in the last propositions, consisting of portions of conical surfaces, is less than four times the greatest circle in the sphere.*

Let *AB • •* •A' • • •B'A be a regular polygon inscribed in a great circle, the number of its sides being a multiple of four.

As before, let *BB'* be drawn sub­tending two sides, and *CC',* • • • Y Y' parallel to *BB'.*



Let *R* be a circle such that the square of its radius is equal to

*AB(BB'+CC'+ • • —FYI"),* so that the surface of the figure in­scribed in the sphere is equal to *R.*

[Prop. 24]

Now

*(BE-FCC+ • • •+YY') :AA'*

***=A'B :AB,*** [Prop. 21] whence *AB(BB'+CC'+ • • •+ YY')* ***=AA' .A'B.***

Hence (radius of *R)2 ---- AA' • A' B*

*<AA'2.*

Therefore the surface of the inscribed figure, or the circle *I?,* is less than four

times the circle *AAIA'M'.*

ON THE SPHERE AND CYLINDER I 19

**PROPOSITION** 26

*The figure inscribed as above in a sphere is equal [in volume] to a cone whose base is a circle equal to the surface of the figure inscribed in the sphere and whose height is equal to the perpendicular drawn from the centre of the sphere to one side of the polygon.*

Suppose, as before, that *AB • • •A' • • •B'A* is the regular polygon inscribed in a great circle, and let *BB', CC', • • •* be joined.

With apex *0* construct cones whose bases are the circles on *BB', CC', • • •* as diameters in planes perpendicular to *A A'.*

M

**C N**

Then *OBAB'* is a solid rhombus, and its volume is

..." equal to a cone whose base is

•

„.• equal to the surface of the cone

**A** *A BB'* and whose height is

equal to the perpendicular

••• from *0* on *AB* [Prop. 18]. Let

•

the length of the perpendicu­lar be p.

Again, if *CB, C'B'* produced meet in *T,* the portion of the solid figure which is described by the revolution of the tri­angle *BOC* about *AA'* is equal to the difference between the rhombi *OCTC'* and *OBTB'*, i.e. to a cone whose base is equal to the surface of the frustum

*BB'C'C* and whose height is p [Prop. 20].

Proceeding in this manner, and adding, we prove that, since cones of equal height are to one another as their bases, the volume of the solid of revolution is equal to a cone with height p and base equal to the sum of the surfaces of the cone *BAB',* the frustum *BB'C'C,* etc., i.e. a cone with height p and base equal to the surface of the solid.

**PROPOSITION** 27

*The figure inscribed in the sphere as before is less than four times the cone whose base is equal to a great circle of the sphere and whose height is equal to the radius of the sphere.*

*By* Prop. 26 the volume of the solid figure is equal to a cone whose base is equal to the surface of the solid and whose height is *p,* the perpendicular from 0 on any side of the polygon. Let *R* be such a cone.

Take also a cone *S* with base equal to the great circle, and height equal to the radius, of the sphere.

Now, since the surface of the inscribed solid is less than four times the great circle [Prop. 25], the base of the cone *R* is less than four times the base of the cone *S.*

Also the height (p) of R is less than the height of *S.*

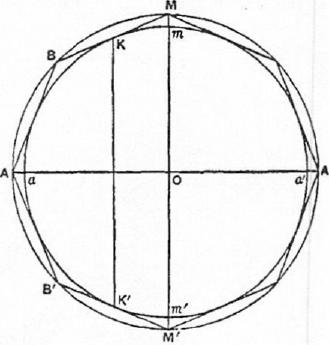
Therefore the volume of *R* is less than four times that of S; and the proposi­tion is proved.

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**PROPOSITION** 28

Let a regular polygon, whose sides are a multiple of four in number, be circum­scribed about a great circle of a given sphere, as *AB • - •A' • • •B'A;* and about the polygon describe another circle, which will therefore have the same centre as the great circle of the sphere. Let AA' bisect the polygon and cut the sphere in a, a'.

If the great circle and the circum­scribed polygon revolve together about *AA',* the great circle will de­scribe the surface of a sphere, the an­gular points of the polygon except *A, A'* will move round the surface of a larger sphere, the points of contact of the sides of the polygon with the great circle of the inner sphere will describe circles on that sphere in planes per­pendicular to *AA',* and the sides of the polygon themselves will describe portions of conical surfaces. *The cir­cumscribed figure will thus be greater than the sphere itself.*



Let any side, as *BAI,* touch the

inner circle in *K,* and let *K'* be the point of contact of the circle with */3/3/'.*

Then the circle described by the revolution of *KK'* about *AA'* is the boun­dary in one plane of two surfaces

1. the surface formed by the revolution of the circular segment *KaK' ,* and
2. the surface formed by the revolution of the part *KB • • •A • • .B'K'* of the polygon.

Now the second surface entirely includes the first, and they are both concave in the same direction;

therefore *[Assumptions,* 4] the second surface is greater than the first. The same is true of the portion of the surface on the opposite side of the circle on *KK'* as diameter.

Hence, adding, we see that *the surface of the figure circumscribed to the given sphere is greater than that of the sphere itself.*

**PROPOSITION** 29

*In a figure circumscribed to a sphere in the manner shown in the previous proposi­tion the surface is equal to a circle the square on whose radius is equal to*

*AB(BB'd-CC'••}- • • .).*

For the figure circumscribed to the sphere is inscribed in a larger sphere, and the proof. of Prop. 24 applies.

**PROPOSITION** 30

*The surface of a figure circumscribed as before about* a *sphere is greater than four*

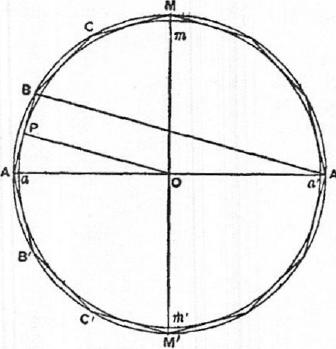
*times the great* circle of *the* sphere.

Let *AB • • •A' • • •B'A* be the regular polygon of 4n sides which by its revolu-

ON THE SPHERE AND CYLINDER I **21**

**tion about *AA'* describes the figure circumscribing the sphere of which ama'm'**

**is a great circle. Suppose aa', *AA'* to** be in one straight line.



Let ***R* be** a circle equal to the **sur­face bf the circumscribed solid. Now**

***(BB' -FCC' + • • -) :AA' -A'B :BA,***

[as in Prop. 21]

so that

***AB(BB' +CC'-1- • • •)= AA' •A'B.* Hence (radius of *R) = VA*** A' • ***A' B***

**[Prop. 29] *>A'B.***

But ***A'B = 20P,* where *P* is the point in which *AB*** touches the circle ama'm'.

Therefore (radius of ***R)>* (diam­eter of** circle ama'm');

whence ***R,*** and therefore the surface of the circumscribed solid, is greater **than** four times the great circle of the given sphere.

**PROPOSITION** 31

*The solid of revolution circumscribed as before about a sphere is equal to a cone whose base is equal to the surface of the solid and whose height is equal to the radius of the* sphere.

The solid is, as before, a solid inscribed in a larger sphere; and, since the perpendicular on any side of the revolving polygon is equal to the radius of the inner sphere, the proposition is identical with Prop. 26.

**COR.** *The solid circumscribed about the smaller sphere is greater than four times the cone whose base is a great circle of the sphere and whose height is equal to the radius of the sphere.*

For, since the surface of the solid is greater than four times the great circle of the inner sphere [Prop. 30], the cone whose base is equal to the surface of the solid and whose height is the radius of the sphere is greater than four times the cone of the same height which has the great circle for base. *[Lemma* **1.]**

**Hence, by the proposition, the volume of the solid is greater than four times the latter cone.**

**PROPOSITION 32**

***If*** *a regular polygon with* 4n *sides be inscribed in a great circle of a sphere, as*

ab • • • • *•b'a,* ***and* a** *similar polygon* ***AB • • •A' • • -B'A*** *be described about the  
great circle, and if the polygons revolve with the great circle about the diameters aa',* ***AA'*** *respectively, so that they describe the surfaces of solid figures inscribed in and circumscribed to the sphere respectively, then*

1. *the surfaces of the circumscribed and inscribed figures are to one another in the duplicate ratio of their sides, and*
2. *the figures themselves [i.e. their volumes] are in the triplicate ratio of their aides.*

**22 ARCHIMEDES**

1. **Let *AA',* ad be in the same straight line, and let *MmOm'M'* be a diam-**

**eter at right angles to them.**

**Join *BE', CC', • • •* and *bb', cc', • • •* which will all be parallel to one another**

**and *MM'.***

**Suppose *R, S* to be circles such that**

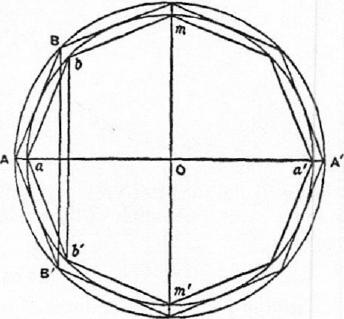
**R4= (surface of circumscribed solid),**

***S=* (surface of inscribed solid).**

**Then (radius of *R)2=AB(Bili-ECC'+ • • •)* [Prop. 29]**

**(radius of *8)2= ab(bb' + cc' + • • •).* [Prop. 24]**

**And, since the polygons are simi­lar, the rectangles in these two equa­tions are similar, and are therefore in the ratio of**



***AB2* : ab2.**

**Hence**

**(surface of circumscribed solid) : (sur­face of inscribed solid) = *AB2 : ab2.***

1. **Take a cone *V* whose base is the circle *R* and whose height is equal to *Oa,* and a cone W whose base is the circle *S* and whose height is equal to the perpendicular from *0* on ab, which we will call *p.***

**Then *V, W* are respectively equal to the volumes of the circumscribed**

**and inscribed figures. [Props. 31, 26]**

**Now, since the polygons are similar,**

***AB :ab=0a :p***

**= (height of cone *V) :* (height of cone W);**

**and, as shown above, the bases of the cones (the circles *R, S) are* in the ratio**

**of *AB2* to *ab2.***

**Therefore *V :W=AB3 :***

**PROPOSITION 33**

***The surface of any sphere is equal to four times the greatest circle in it.***

**Let *C* be a circle equal to four times the great circle.**

**Then, if *C* is not equal to the surface of the sphere, it must either be less or**

**greater.**

**I. Suppose *C* less than the surface of the sphere.**

**It is then possible to find two lines -y, of which # is the greater, such that**

**# : y < (surface of sphere) : *C.* [Prop. 2]  
Take such lines, and let S be a mean proportional between them.**

**Suppose similar regular polygons with 4n sides circumscribed about and inscribed in a great circle such that the ratio of their sides is less than the**

**ratio *: S.* [Prop. 3]  
Let the polygons with the circle revolve together about a diameter common to all, describing solids of revolution as before.**

ON THE SPHERE AND CYLINDER I 23

Then (surface of outer solid) : (surface of inner solid)

= (side of outer)2 : (side of inner)= [Prop. 32)

< : 62, or *:* -y

< (surface of sphere) : *C,* a *fortiori.*

M

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But this is impossible, since the surface of the circumscribed solid is greater

than that of the sphere [Prop. 28], while the surface of the inscribed solid is less

than *C* [Prop. 25].

Therefore C is not less than the surface of the sphere.

II. Suppose *C* greater than the surface of the sphere.

Take lines $, -y, of which $ is the greater, such that

: y <C : (surface of sphere).

Circumscribe and inscribe to the great circle similar regular polygons, as be-

fore, such that their sides are in a ratio less than that of $ to *6,* and suppose

solids of revolution generated in the usual manner.

Then, in this case,

(surface of circumscribed solid) : (surface of inscribed solid)

*<C* : (surface of sphere).

But this is impossible, because the surface of the circumscribed solid is

greater than *C* [Prop. 30], while the surface of the inscribed solid is less than

that of the sphere [Prop. 23].

Thus *C* is not greater than the surface of the sphere.

Therefore, since it is neither greater nor less, *C* is equal to the surface of the

sphere.

**PROPOSITION** 34

*Any sphere is equal to four times the cone which has it bow equal to the greatest*

*circle in the sphere and its height equal to the radius of the sphere.*

Let the sphere be that of which *ama'm'* is a great oircle.

If now the sphere is not equal to four times the cone described, it is either

greater or less.

I. If possible, let the sphere be greater than four times the cone.

Suppose *V* to be a cone whose base is equal to four times the great circle and

whose height is equal to the radius of the sphere.

24 ARCHIMEDES

Then, by hypothesis, the sphere is greater than *V;* and two lines *0, y* can be found (of which 0 is the greater) such that

*.43 : 'y <* (volume of sphere) : *V.*

Between *(3* and *y* pl:aee two arithmetic means 8, **e.**

As before, let similar regular polygons with sides 4n in number be circum­scribed about and inscribed in the great circle, such that their sides are in a ratio less than 0 : S.

Imagine the diameter aa' of the circle to be in the same straight line with a diameter of both polygons, and imagine the latter to revolve with the circle about cc', describing the surfaces of two solids of revolution. The volumes of

these solids are therefore in the triplicate ratio of their sides. [Prop. 32]

Thus (vol. of outer solid) : (vol. of inscribed solid)

< : 33, by hypothesis,

<13 : y, a *fortiori* (since $ : *7> j3" : 0),*

< (volume of sphere) : *V,* a *fortiori.*

M

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| *f3* 8 c *7* | | | | |

But this is impossible, since the volume of the circumscribed solid is greater

than that of the sphere [Prop. 28], while the volume of the inscribed solid is

less than V [Prop. 27].

Hence the sphere is not greater than *V,* or four times the cone described in

the enunciation.

II. If possible, let the sphere be less than *V.*

In this case we take *0, 7 (0* being the greater) such that

,t3 : y *< V :* (volume of sphere).

The rest of the construction and proof proceeding as before, we have finally

(volume of outer solid) : (volume of inscribed solid)

< *V* : (volume of sphere).

But this is impossible, because the volume of the outer solid is greater than

V [Prop. 31, Cor.], and the volume of the inscribed solid is less than the volume

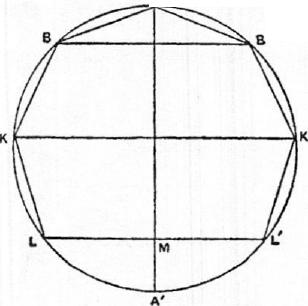
of the sphere.

Hence the sphere is not less than *V.*

Since then the sphere is neither less nor greater than *V,* it is equal to *V,* or to

four times the cone described in the enunciation.

ON THE SPHERE AND CYLINDER I 25



COR. From what has been proved it follows that *every cylinder whose base is the greatest circle in a sphere and whose height is equal to the diameter of the sphere*

*is of the sphere,* and *its surface together with its bases is of the surface of the  
sphere.*

For the cylinder is three times the cone with the same base and height [Eucl. xi" 10], i.e. six times the cone with the same base and with height equal to the radius of the sphere.

But the sphere is four times the latter cone [Prop. 34]. Therefore the cylinder is of the sphere.

Again, the surface of a cylinder (excluding the bases) is equal to a circle whose radius is a mean proportional between the height of the cylinder and the diameter of its base [Prop. 13].

In this case the height is equal to the diameter of the base and therefore the circle is that whose radius is the diameter of the sphere, or a circle equal to four times the great circle of the sphere.

Therefore the surface of the cylinder with the bases is equal to six times the great circle.

And the surface of the sphere is four times the great circle [Prop. 33] ; whence (surface of cylinder with bases) =3- .(surface of sphere).

PROPOSITION 35

*If in* a *segment of a circle LA L' (where A is the middle point of the arc) a polygon LK • • •A • • •K'L' be inscribed of which LL' is one side, while the other sides are* 2n *in number and all equal, and if the polygon revolve with the segment about the diameter AM, generating a solid figure inscribed in* a *segment of a sphere, then the surface of the inscribed solid is equal to a circle the square on whose radius is equal to the rectangle*

*'*

*AB (BB' +CC+ • • KK' -FLL)*

2

The surface of the inscribed figure is

**A** made up of portions of surfaces of cones.  
If we take these successively, the surface of the cone *BAB' is* equal to a circle whose radius is

*\/AB • -1-BB'.* [Prop. 14]  
The surface of the frustum of a cone *BCC'B'* is equal t o a circle whose radius is

*AB •BB' BB'-}'CC'. •* [P 2 ' rop. 16]

and so on.

Proceeding in this way and adding, we find, since circles are to one another as the squares of their radii, that the surface of the inscribed figure is equal to a circle whose radius is

*±L2L).*

*\IAB*

26 ARCIIIMEDES

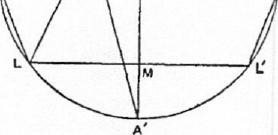
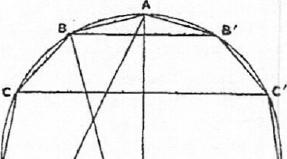
**PROPOSITION** 36

*The surface of the figure inscribed as before in the segment of a sphere is less than that of the segment of the sphere.*

This is clear, because the circular base of the segment is a common boundary of each of two surfaces, of which one, the segment, includes the other, the solid, while both are concave in the same direction *[Assumptions, 4].*

**PROPOSITION** 37

*Dhe surface of the solid figure inscribed in the segment of the sphere by the revolu­tion of LK • • •A • • •K'L' about* AM *is less than a circle with radius equal to AL.*



Let the diameter *AM* meet the circle of which *LALl•* is a **8egrtftt** again in *A'.* Join *A'B.*

As in Prop. 35, the surf ace of the inscribed solid is equal to a circle the square on whose radius is

*AB(BB1-1-CC'l- • • :+KK'-FLM).* But this rectangle

*= A'B •AM* [Prop. 22] *<A'A •AM*

*<AL2.*

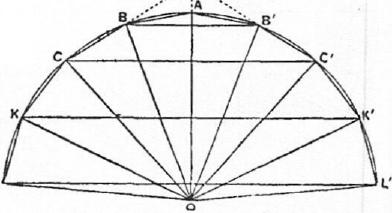
Hence the surface of the inscribed solid is less than the circle whose radius *is AL.*

**PROPOSITION** 38

*The solid figure described as before in* a *segment of* a *sphere less than a hemisphere, together with the cone whose base is the base of the segment and whose apex is the*

*centre of the sphere, is equal to a cone whose base is equal to the surface of the inscribed solid and whose height is equal to the perpendicular from the centre of the sphere on* any *side of the polygon.*

Let 0 be the centre of the sphere, and *p* the length of the perpendicular from 0 on *AB.*



Suppose cones described with 0 as apex, and with the circles on *BB', CC', • • • as* diameters as bases.

Then the rhombus *OBAB'* is equal to a cone whose base is equal to the surface of the cone *BAB',* and whose

height is p. [Prop. 18]

Again, if *CB, C'B'* meet in ***T,*** the solid described by the triangle *BOC* as the polygon revolves about *AO* is the difference between the rhombi *OCTC'* and *OB7'8', and is* therefore equal to a cone whose base is equal to the surface of

ON THE SPHERE AND CYLINDER I 27

the frustum *BCC' B'* and whose height is p. [Prop. 20]

Similarly for the part of the solid described by the triangle *COD* as the poly­gon revolves; and so on.

Hence, by addition, the solid figure inscribed in the segment together with the cone *OLL' is* equal to a cone whose base is the surface of the inscribed solid and whose height is p.

**Cox.** *The cone whose base is a circle with radius equal to AL and whose height is equal to the radius of the sphere is greater than the sum of the inscribed solid and the cone OLL'.*

For, by the proposition, the inscribed solid together with the cone *OLL'* is equal to a cone with base equal to the surface of the solid and with height p.

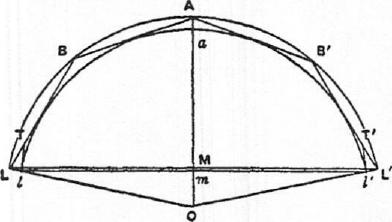
This latter cone is less than a cone with height equal to OA and with base equal to the circle whose radius *is AL,* because the height p is less than OA, while the surface of the solid is less than a circle with radius *AL.* [Prop. 37]

PROPOSITION 39

Let *lal'* be a segment of a great circle of a sphere, being less than a semicircle. Let *0* be the centre of the sphere, and join *01, 01'.* Suppose a polygon circum­scribed about the sector *Olal'* such that its sides, excluding the two radii, are 2n in number and all equal, as *LK, • • .BA, AB', • • •K'L';* and let OA be that radius of the great circle which bisects the segment *lal'.*

The circle circumscribing the polygon will then have the same centre *0* as the given great circle.

Now suppose the polygon and the two circles to revolve together about *OA.* The two circles will describe spheres, the angular points except A will describe circles on the outer sphere, with diameters *BB'* etc., the points of contact of the sides with the inner segment will describe circles on the inner sphere, the sides themselves will describe the surfaces of cones or frusta of cones, and the whole figure circumscribed to the segment of the inner sphere by the revolu­tion of the equal sides of the polygon will have for its base the circle on *LL'* as diameter.



*The surface of the solid figure so circumscribed about the sector of the sphere [excluding its base] will be greater than that of the segment of the sphere whose base is the circle on 11' as diameter.*

For draw the tangents *1T, l'T'* to the inner segment at *1, 1'.* These with the sides of the polygon will describe by their revolution a solid whose surface is greater than that of the segment *[Assumptions,* 4].

But the surface described by the revolution of *1T is* less than that described by the revolution of *LT,* since the angle *T1L* is a right angle, and therefore *LT>1T.*

Hence, *a fortiori,* the surface described by *LK • • •A • • •K'L'* is greater than that of the segment.

**28 ARCHIMEDES**

**Co** *1. The surface of the figure so described about the sector of the sphere is equal*

*to a circle the square on whose radius is equal to the rectangle*

*AB (13131-1-CC'+ • • • +KK' +ILL%*

For the circumscribed figure is inscribed in the outer sphere, and the proof

of Prop. 35 therefore applies.

**PROPOSITION** 40

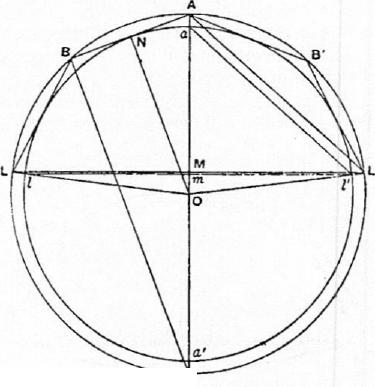
*The surface of the figure circumscribed to the sector as before is greater than a circle whose radius is equal to al.*

Let the diameter *Aa0* meet the great circle and the circle circumscribing the revolving polygon again in *a', A'.* Join *A'B,* and let *ON* be drawn to *N,* the point of contact of *AB* with the inner circle.

Now, by Prop. 39, Cor., the surface of the solid figure circumscribed to the sector *OlAl'* is equal to a circle the square on whose radius is equal to the rectangle

|  |  |
| --- | --- |
| *AB (BB'A-CC1+ • • •-f-KK'* | *LL' +T).* |

But this rectangle is equal to *A'B • AM* [as in Prop. 22].



Next, since *AL', al'* are paral­lel, the triangles *A M L' , cue* are similar. And *AL' > al';* therefore *AM* > am.

Also *A'B =20N =* aa'.

Therefore *A'B •AM>* am -ad

*>al'2.*

Hence the surface of the solid figure circumscribed to the sector is greater than a circle whose radius is equal to *al',* or *al.*

**CoR. 1.** *The volume of the figure circumscribed about the sector to­gether with the cone whose apex is° and base the circle on LL' as diam­eter, is equal to the volume of a cone*

*whose base is equal to the surface of the circumscribed figure and whose height is ON.*

For the figure is inscribed in the outer sphere which has the same centre as the inner. Hence the proof of Prop. 38 applies.

**CoR. 2.** *The volume of the circumscribed figure with the cone OLL' is greater than the cone whose base is a circle with radius equal to al and whose height is equal to the radius (Oa) of the inner sphere.*

For the volume of the figure with the cone *OLL'* is equal to a cone whose base is equal to the surface of the figure and whose height is equal to *ON.*

And the surface of the figure is greater than a circle with radius equal to *al* [Prop. 40], while the heights *Oa, ON* are equal.

ON THE SPHERE AND CYLINDER I **29**

**PROPOSITION** 41

Let *lar* be a segment of a great circle of a sphere Ivhich is less than a semicircle.

Suppose a polygon inscribed in the sector *Olal'* such that the sides *lk, • • •ba, all, ...el'* are 2n in number and all equal. Let a similar polygon be circum­scribed about the sector so that its sides are parallel to those of the first poly­gon; and draw the circle circumscribing the outer polygon.

Now let the polygons and circles revolve together about *Oa A,* the radius bisecting the segment *lar.*

Then *(1) the surfaces of the outer and inner solids of revolution so described are in the ratio of AB2 to ab2, and* (2) *their volumes together with the corresponding cones with the* same *base and with apex 0 in each* case are *as AB3 to ab3.*

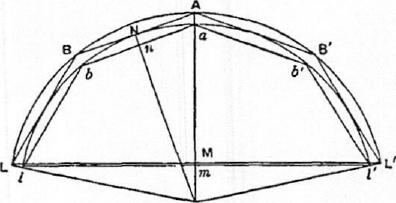
(1) For the surfaces are equal to circles the squares on whose radii are equal respectively to

*AB(B.B' -1-CC'+ • • •+KIC' LL'* [Prop. 39, Cor.]

**and** *ab (bb' +cc' ± • • •-f-kk'+—* **2** *).* [Prop. 35]

But these rectangles are in the ratio of *AB2* to ab2. Therefore so are the surfaces.

**(2)** Let *OnN* be drawn per­pendicular to *ab* and *AB;* and suppose the circles which are equal to the surfaces of the outer and inner solids of revo­lution to be denoted by *S, s* respectively.



Now the volume of the cir­cumscribed solid together with the cone *OLL'* is equal to a cone whose base is *S* and whose height *is ON* [Prop. 40, Cor. **1].**

**And** the volume of the inscribed figure with the cone *Ole is* equal to **a** cone with base *s* and height *On* [Prop. 38].

But S *s* = *AB2: ab2,*

and *ON : On= AB : ab.*

Therefore the volume of the circumscribed solid together with the cone *OLL' is* to the volume of the inscribed solid together with the cone *011'* as AB3 is to ab3 *[Lemma* 5].

**PROPOSITION** 42

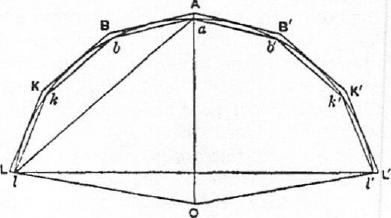
*If lal' be a segment of* a *sphere less than a hemisphere and Oa the radius perpen­dicular to the base of the segment, the surface of the segment is equal to a circle whose radius is equal to al.*

Let *R* be a circle whose radius is equal to *al.* Then the surface of the segment, which we will call *S,* must, if it be not equal to *R,* be either greater or less **than** *R.*

30 ARCHIMEDES

1. Suppose, if possible, S>I?.

Let *lal'* be a segment of a great circle which is less than a semicircle. Join *01, 01',* and let similar polygons with 2n equal sides be circum­scribed and inscribed to the sector, as in the previous prop­ositions, but such that (circumscribed polygon) : (in­scribed polygon) *<45 : I?.*



[Prop. 6] Let the polygons now revolve with the segment about OaA, generating solids of revolution circumscribed and inscribed to the segment of the sphere.

Then

(surface of outer solid) : (surface of inner solid)

= *AB2* : ab2 [Prop. 41]

= (circumscribed polygon) : (inscribed polygon)

*<S :* R, by hypothesis.

But the surface of the outer solid is greater than *S* [Prop. 39].

Therefore the surface of the inner solid is greater than *I?;* which is impossible,

by Prop. 37.

1. Suppose, if possible, *8 <R.*

In this case we circumscribe and inscribe polygons such that their ratio is

less than *R : S;* and we arrive at the result that

(surface of outer solid) : (surface of inner solid)

*<I? : S.*

But the surface of the outer solid is greater than R [Prop. 40]. Therefore the

surface of the inner solid is greater than *S :* which is impossible [Prop. 36].

Hence, since *S* is neither greater nor less than *R,*

*S=R.*

**PROPOSITION** 43

*Even if the segment of the sphere is greater than a hemisphere, its surface is still*

*equal to a circle whose radius is equal to al.*

For let *lal'a'* be a great circle of the sphere, aa' being the diameter perpen-

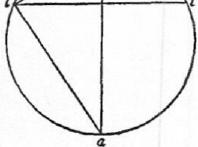
dicular to *ii';* and let *la'l'* be a segment less than a

semicircle.

Then, by Prop. 42, the surface of the segment *la'l'*

of the sphere is equal to a circle with radius equal to

*a'1.*



Also the surface of the whole sphere is equal to a

circle with radius equal to aa' [Prop. 33].

But *aa12 — a'12 = al2,* and circles are to one another

as the squares on their radii.

Therefore the surface of the segment *lal',* being

the difference between the surfaces of the sphere and

of *la'l',* is equal to a circle with radius equal to *al.*

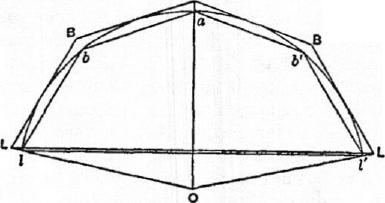
ON THE SPHERE AND CYLINDER I 31

**PROPOSITION** 44

*The volume of any sector of a sphere is equal to a cone whose base is equal to the surface of the segment of the sphere included in the sector, and whose height is equal to the radius of the sphere.*

Let ***R*** be a cone whose base is equal to the surface of the segment *tat'* of a sphere and whose height is equal to the radius of the sphere; and let *S* be the volume of the sector *Olal'.*

p



**a**

Then, if *S* is not equal to ***R,*** it must be either greater or less.

I. Suppose, if possible, that *S>* ***R.***

Find two straight lines *fi, 7,* of which /3 is the greater, such that

***t3:7<S:R;***

and let (5, *e* **be** two arithmetic means between /3, 7.

Let *tat'* be a segment of a great circle of the sphere. Join ***01,*** 01', and let similar poly-

7  gons with 2n equal sides be  
circumscribed and inscribed to the sector of the circle as before, but such that their sides are in a ratio less than ***fi : S. [Prop.*** 4].

Then let the two polygons revolve with the segment about *OaA,* generating two solids of revolution.

Denoting the volumes of these solids by ***V,*** *v* respectively, we have

(V-1-cone ***OLL') :*** (v+cone ***OW) =AB3 : ab3*** {Prop. 41]

<9:a3

< 13 : **7, *a fortiori,***

***<S : R,*** by hypothesis.

Now (V-1-cone OLL') *>S.*

Therefore also (v+cone ***011')> R.***

But this is impossible, by Prop. 38, Cor. combined with Props. 42, 43.

Hence *S>* ***R.***

***II.*** Suppose, if possible, that ***S <R.***

In this case we take 0, **y** such that

/3. .y *<R* :8,

and the rest of the construction proceeds as before.

We thus obtain the relation

(V-I-cone ***OLL') :*** (v- -cone ***011') <R : S.***

Now (v-}-cone ***011') <S.***

Therefore (V-}-cone ***OLL') <R;***

which is impossible, by Prop. 40, Cor. 2 combined with Props. 42, 43.

Since then *S* is neither greater nor less than ***R,***

***S=R.***

**ON THE SPHERE AND CYLINDER  
BOOK TWO**

ARCHIMEDES to Dositheus greeting.

"On a former occasion you asked me to write out the proofs of the problems t he enunciations of which I had myself sent to Conon. In point of fact they depend for the most part on the theorems of which I have already sent. you the demonstrations, namely (1) that the surface of any sphere is four times the greatest circle in the sphere, (2) that the surface of any segment of a sphere is equal to a circle whose radius is equal to the straight line drawn from the ver­tex of the segment to the circumference of its base, (3) that the cylinder whose base is the greatest circle in any sphere and whose height is equal to the diam­eter of the sphere is itself in magnitude half as large again as the sphere, while its surface [including the two bases] is half as large again as the surface of the sphere, and (4) that any solid sector is equal to a cone whose base is the circle which is equal to the surface of the segment of the sphere included in the sec­tor, and whose height is equal to the radius of the sphere. Such then of the theorems and problems as depend on these theorems I have written out in the book which I send herewith; those which are discovered by means of a different sort of investigation, those namely which relate to spirals and the conoids, I will endeavour to send you soon.

"The first of the problems was as follows: *Given a sphere, to find a plane area equal to the surface of the sphere.*

"The solution of this is obvious from the theorems aforesaid. For four times the greatest circle in the sphere is both a plane area and equal to the surface of the sphere.

"The second problem was the following."

PROPOSITION 1 (PROBLEM)

*Given a cone or a cylinder, to find a sphere equal to the cone or to the cylinder.* If *V* be the given cone or cylinder, we can make a cylinder equal to *W.* Let this cylinder be the cylinder whose base is the circle on *AB* as diameter and whose height is *OD.*

Now, if we could make another cylinder, equal to the cylinder *(OD)* but such that its height is equal to the diameter of its base, the problem would be solved, because this latter cylinder would be equal to IV, and the sphere whose diam­eter is equal to the height (or to the diameter of the base) of the same cylinder would then be the sphere required [I. 34, Cor.].

Suppose the problem solved, and let the cylinder (CG) be equal to the cylin­der *(OD),* while *EF,* the diameter of the base, is equal to the height *CG.*

32

ON THE SPHERE AND CYLINDER II 33

Then, since in equal cylinders the heights and bases are reciprocally pro-

portional,

*AB2 : EF2=CG :OD*

*=EF :OD.* (1)

M

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Suppose *MN* to be such a line that

' *EF2=AB MIN.* (2)

Hence *AB : EF=EF : MN,*

and, combining (1) and (2), we have

*AB : MN =EF : OD,*

or *AB : EP' =MN :OD.*

Therefore *AB :EF=EF : MN =MN : OD,*

and *EF, MN are two mean proportionals between AB, OD.*

The synthesis of the problem is therefore as follows. Take two mean pro-

portionals *EF, MN* between *AB* and *OD,* and describe a cylinder whose base

is a circle on *EF* as diameter and whose height *CG* is equal to *EF.*

Then, since

*AB : EF = EF :MN=MN : OD,*

*EF2 = AB •MN,*

and therefore *AB2* : *EF2=AB : MN*

*=EF :OD*

*=CG : OD;*

whence the bases of the two cylinders *(OD), (CG)* are reciprocally proportional

to their heights.

Therefore the cylinders are equal, and it follows that

cylinder *(CG) =V.*

The sphere on *EF* as diameter is therefore the sphere required, being equal

to *V.*

**PROPOSITION** 2

*If BAB' be a segment of a sphere, B13' a diameter of the base of the segment, and 0 the centre of the sphere, and if AA' be the diameter of the sphere bisecting BB' in M, then the volume of the segment is equal to that of a cone whose base is the same as that of the segment and whose height is h, where*

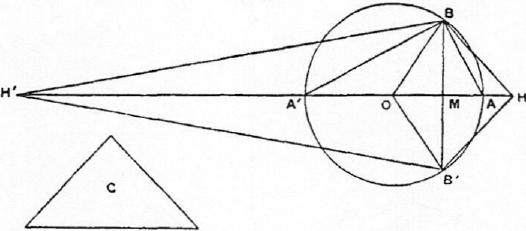
*h :AM=0A'±A'M : A'M.*

Measure *MI?* along *MA* equal to *h,* and *MH'* along *MA'* equal to *h',* where

*h' : A'M=OA+AM :AM.*

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Suppose the three cones constructed which have ***0, H, 11'*** for their apices and the base ***(BB')*** of the segment for their common base. Join *AB,* ***A'B.***



Let *C* be a cone whose base is equal to the surface of the segment ***BAB'*** of the sphere, i.e. to a circle with radius equal to ***AB* [I.** 42], and whose height is equal to ***OA.***

Then the cone *C* is equal to the solid sector ***OBAB'* [I.** 44].

Now, since ***HM :MA=0A'+A'111 :A'M,***

***dividendo, HA :AM=0A : A'M,***

and, alternately, ***HA : AO =AM : MA',***

so that

***HO : OA =AA' : A'M***

***=AB2 : BM2***

= (base of cone *C)* : (circle on ***BB'*** as diameter).

But ***OA*** is equal to the height of the cone *C;* therefore, since cones are equal if

their bases and heights are reciprocally proportional, it follows that the cone *C*

(or the solid sector ***OBAB')*** is equal to a cone whose base is the circle on ***BB'*** as

diameter and whose heigkit is equal to ***OH.***

And this latter cone is equal to the sum of two others having the same base

and with heights ***OM, M11, i.e. \*to*** the solid rhombus ***01311.8'.***

Hence the sector ***OBAB'*** is equal to the rhombus ***OBHB'.***

Taking away the common part, the cone ***OBB',***

the segment ***BAB' =the*** cone ***HBB'.***

Similarly, by the same method, we can prove that

the segment BA'B' = the cone ***H'BB'.***

***Alternative proof of the latter property.***

***SupPw*** 4 to be a cone whose base is equal to the surface of the whole sphere

and whose height is equal to ***OA.***

Thus ***D*** is equal to the volume of the sphere. [1. 33, 34]

Now, since ***OA'+A'M :A'M=HM : MA,***

***dividendo*** and aiternando, as before,

OA : ***AH=A'M :MA.***

Again, since ***H'M : MA' =0A+AM : AM,***

***WA' :OA= A'M :MA***

***=OA : AH,*** from above.

***Componendo, : OA =OH : HA, (1)***

Alternately. ***H'0 :011=0A : A H****,* **(2)**

ON THE SPHERE AND CYLINDER II 35

and, *componendo, : HO =OH : HA,*

*=H'0 : OA,* from (1),

whence *HH' • OA =H'0 • OH.* (3)

Next, since *H'0 : OH = OA : AH, by* (2),

*=A'M : MA,*

*(11/0+0H)2 : H'0 • OH = (A'M-FMA)2 : A'M MA,*

whence, by means of (3),

*HH'2 :HH' • OA* =AA" : A'M•• *MA,*

or *HH' :OA = AZ112* **:** *BAP.*

Now the cone *D,* which is equal to the sphere, has for its base a circle whose

radius is equal to *AA',* and for its height a line equal to *OA.*

Hence this cone *D* is equal to a cone whose base is the circle on *BR'* as diam-

eter and whose height is equal to *HH';*

therefore the cone *D=* the rhombus *HRIPB',*

or the rhombus *HBH'B' = the* sphere.

But the segment *BAB' = the* cone *H BB' ;*

therefore the remaining segment *BA'B' =* the cone *H'BB'.*

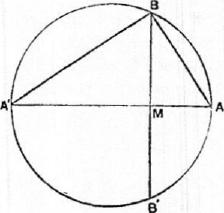
Cox. *The segment BAB' is to a cone with the same base and equal height in the*

*ratio of OA'-i-A'M to A'M.*

PROPOSITION 3 (PRoBLI•:1i)

*To cut a given sphere by a plane so that the surfaces of the segments may have to one another* a *given ratio.*

Suppose the problem solved. Let *AA'* be a diameter of a great circle of the sphere, and suppose that a plane per­pendicular to *AA'* cuts the plane of the great circle in the straight line *BB',* and *AA'* in *M,* and that it divides the sphere so that the surface of the seg­ment *BAB'* has to the surface of the



**H K**

segment *BA'B'* the given ratio.

Now these surfaces are respectively

equal to circles with radii equal to *AB,*

*A'B [I.* 42, 43].

Hence the ratio *AB2* : A'B2 is equal to

the given ratio, i.e. *AM is* to *MA'* in the

given ratio.

Accordingly the synthesis proceeds as follows.

If *H : K* be the given ratio, divide *AA'* in *M* so that

*AM : MA' =H : K.*

Then *AM : MA' = AB2 : A'B2*

= (circle with radius *AB) :* (circle with radius *A'B)*

= (surface of segment *BAB') :* (surface of segment *BA'B').*

Thus the ratio of the surfaces of the segments is equal to the ratio *H : K.*

PROPOSITION 4 (PROBLEM)

*To cut a given sphere by a plane so that the volumes of the segments are to one*

*another in* a *given ratio.*

Suppose the problem solved, and let the required plane cut the great circle

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*ABA'* at right angles in the line *BB'.* Let *AA'* be that diameter of the great circle which bisects *BB'* at right angles (in *M),* and let *0* be the centre of the sphere.

|  |  |
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Take *H* on *OA* produced, and *H'* on *OA'* produced, such that

*OA'+A'M : A'M=HM : MA,* **(1)**

and *OA+AM : AM =H'M : MA'.* **(2)**

Join *BH, B'H, BH', B'H'.*

Then the cones *HBB', H'BB'* are respectively equal to the segments *BAB',*

*BA'B'* of the sphere [Prop. 2].

Hence the ratio of the cones, and therefore of their altitudes, is given, i.e.

*HM H'M =* the given ratio. (3)

We have now three equations (1), (2), (3), in which there appear three as yet undetermined points *M, H, H';* and it is first necessary to find, by means of them, another equation in which only one of these points (M) appears, i.e. we have, so to speak, to *eliminate H, H'.*

Now, from (3), it is clear that *IIH' : H'M* is also a given ratio; and Archi­medes' method of elimination is, *first,* to find values for each of the ratios *A'H' : H'M* and *HIP : H'A'* which are alike independent of *II, H',* and then, *secondly,* to equate the ratio compounded of these two ratios to the known value of the ratio *IIH'*

1. To find such a value for *A'H' : H'M.* It is at once clear from equation (2) above that

*A'H' : M =DA : OA+AM.* **(4)**

1. To find such a value for *HH' : A'H'.*

From (1) we derive

*A'M :MA=0A'+A'M :HM*

*=OA' : AH;* (5)

and, from (2), *A'M : MA = :0A+AM*

*A'H' :OA.* (6)

Thus *HA : AO =OA' :*

whence *OH : OA' =OH' : A'H',*

or *OH :OH' =OA' : A'H'.*

It follows that

*HH' : OH' =OH' : A'H',*

or *HH' •H'A'=011'2.*

Therefore *HH` : H'A' =OH'2 :H'A'2*

*=AA/2 : A'M2,* by means of (6)

1. To express the ratios *A'H' : H'M* and *HIP : H'M* more simply we make

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the following construction. Produce *OA* to *D* so that *OA = AD. (D* will lie be­yond ***H,*** for *A'AI > M A,* and therefore, by (5), *OA> AH.)*

Then *A'H' : H'M =OA : OA +AM*

*=AD :DM.* (7)

Now divide *AD* at *E so* that

*HH' : H'M =AD : DE.* (8)  
Thus, using equations (8), (7) and the value of *HH' : H'A'* above found, we have

*AD :DE=HH' : H'M*

*= : H'A') •(A'H' 11'M)  
= (AA'2 : A'M2) •(AD : DM).*

But AD : *DE= (DM : DE) •(AD :* DM).

Therefore *MD : DE = AA" : A'M2.* (9)  
And *D is* given, since AD *=OA.* Also *AD : DE* (being equal to ***HH' : H'M)*** is a given ratio. Therefore *DE* is given.

*Hence the problem reduces itself to the problem of dividing A' D into two parts at Al so that*

*MD :* (a given length) = (a given area) : *A'M2.*

Archimedes adds: "If the problem is propounded in this general form, it requires a *otoptal.tin* [i.e. it is necessary to investigate the limits of possibility], but, if there be added the.conditions subsisting in the present case, it does not require a ocoptabios."

In the present case the problem is:

*Given a straight line* A'A *produced to D so that A'A = 2AD, and given a point E on AD, to cut AA' in a point M so that*

*AA'2 : A'Al2=MD : DE.*

"And the analysis and synthesis of both problems will be given at the end."'

The synthesis of the main problem will be as follows. Let *R : S* be the given ratio, *R* being less than *S. AA'* being a diameter of a great circle, and *0* the centre, produce *OA* to *D* so that *OA =AD,* and divide AD in *E* so that

*AE : ED = R : 8.*

Then cut *AA' in Al* so that

MD *DE = AA'2 : A' Al2.*

Through *M* erect a plane perpendicular to *AA';* this plane will then divide

the sphere into segments which will be to one another as *R* to *S.*

'Fake *II* on *A'A* produced, and *H'* on *AA'* produced, so that

OA'-1-A/M A'M=HM : *MA, (1)*

*OA + AM : AM = H' AI : MA'.* **(2)**

We have then to show that

*JIM : M H' =R :S,* or *AE : ED.*

(a) We first find the value of *: H'A'* as follows.

As was shown in the analysis ***(b),***

*HIP •11' A' =01112,*

or *IIH' : H'A' =OH'2 : H'A'2*

*=AA" : A'M2*

*=AID : DE,* by construction.

'As Archimedes' commentator, Eutocius, notes: "... we do not find the promise kept in any of the copies." Sir Thomas Heath's translation of Eutocius' note on the matter, along with the solutions of Dionysodorus and Diodes, is omitted from this edition.—ED.

|  |  |
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| **38** ARCHIMEDES  (13) Next we have  *H'A' : 111 M =OA : OA+ AM*  *=AD :DM.*  Therefore *HH' : H'M = (HH' : H'A') •(H` A' : H'M)*  *= (MD : DE) •(AD : DM) =AD : DE,*  whence *HM : MH' =AE :ED*  *=1? : S.* | Q. E. D. |

PROPOSITION 5 (PROBLEM)

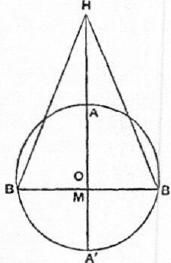
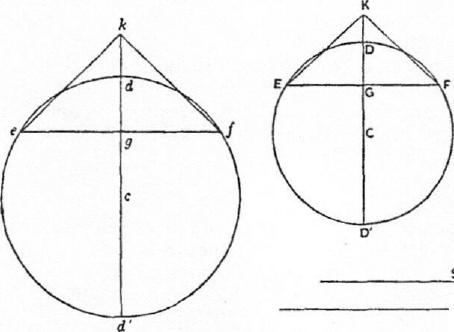
*To construct a segment of a sphere similar to one segment and equal* in *volume to another.*

Let *ABB'* be one segment whose vertex is *A* and whose base is the circle on *BB'* as diameter; and let *DEF* be another segment whose vertex is *D* and whose base is the circle on *EF* as diameter. Let *AA', DD'* be diameters of the great circles passing through *BB', EF* respectively, and let 0, *C* be the respec­tive centres of the spheres.

Suppose it required to draw a segment similar to *DEF* and equal in volume to *ABB'.*

*Analysis.* Suppose the problem solved, and let *def* be the required segment, *d* being the vertex and *ef* the diameter of the base. Let *dd'* be the diameter of the sphere which bisects *ef* at right angles, *c* the centre of the sphere.

Let M, G, g be the points where ***BB', EF,*** ef are bisected at right angles by



R

*AA', DD', dd'* respectively, and produce *OA, CD, cd* respectively to ***H, K,*** *k,*

so that

*OA' -1-A1M : A'M =HM :MA)*

*:D'G=KG :GD*

*cd'-Fd'g : d'g --kg : gd*

and suppose cones formed with vertices *H, K,* k and with the same bases as the

respective segments. The cones will then be equal to the segments respectively

[Prop. 2].

Therefore, by hypothesis,

**ON THE SPHERE AND CYLINDER II 39**

**the *cone* the cone *kef.***

**Hence**

**(circle on diameter *BB') :* (circle on diameter.ef)..-- *kg : HM,***

***so* that *BB'2 : ef2=kg : HM (1)*But, since the segments *DEF, def are* similar, so are the cones *KEF, kef.***

**Therefore *KG : EF = kg : ef.***

**And the ratio *KG EF* is given. Therefore the ratio *kg : ef* is given.**

**Suppose a length *R* taken such that**

***kg : ef =HM : R.* (2)**

**Thus R is given.**

**Again, since *kg : HM =BB'2 : eP=ef : R,* by (1) and (2), suppose a length *S***

**taken such that**

***ef2 = BB' .8,***

**or *BB'2* : *ef2 =BB' : S.***

**Thus *BB' : ef =ef : S=8 : R,***

**and *ef, S are two mean proportionals in continued proportion between BB', R.***

***Synthesis.* Let *ABB', DEF* be great circles, *AA', DD'* the diameters bisecting**

***BB', EF* at right angles in *M, G* respectively, and *0, C* the centres.**

**Take *H, K* in the same way as before, and construct the cones *HBB', KEF,***

**which are therefore equal to the respective segments *ABB', DEF.***

**Let it be a straight line such that**

***KG :EF=HM :R,***

**and between *BB', R* take two mean proportionals *ef, S.***

**On *ef* as base describe a segment of a circle with vertex *d* and similar to the**

**segment of a circle *DEF.* Complete the circle, and let *dd'* be the diameter**

**through *d,* and *c* the centre. Conceive a sphere constructed of which *def is* a**

**great circle, and through *ef* draw a plane at right angles to *dd'.***

**Then shall *def* be the required segment of a sphere.**

**For the segments *DEF, def* of the spheres are similar, like the circular seg-**

**ments *DEF, def.***

**Produce *cd* to *k* so that**

***cd'+d'g : d'g =kg : gd.***

**The cones *KEF, kef* are then similar.**

**Therefore *kg : ef = KG : EF= HM : R,***

**whence *kg :HM=ef :R.***

**But, since *BB', ef, S, R* are. hi continued proportion,**

***BB" : efl =BB' : S***

***=ef:R***

***=kg :HM.***

**Thus the *bases* of the cones *HBB', kef are* reciprocally proportional to their**

**heights. The cones are therefore equal, and *def* is the segment required, being**

**equal in volume to the cone *kef.* [Prop. 2]**

**PROPOSITION 6 (PROBLEM)**

***Given two segments of spheres, to find a third segment of a sphere similar to one of the given segments and having its surface equal to that of the other.***

**Let *ABB'* be the segment to whose surface the surface of the required seg. ment is to be equal, *ABA'B'* the great circle whose plane cuts the plan© of the**

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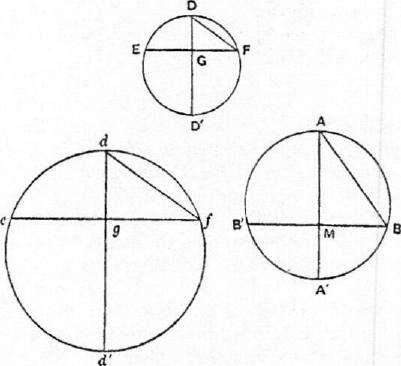
**base of the segment *ABB'* at right angles in *BB'.* Let *AA'* be the diameter which bisects *BB'* at right angles.**

**Let *DEF* be the segment to which the required segment is to be similar, *DED'F* the great circle cutting the base of the segment at right angles in *EF.* Let *DD'* be the diameter bisecting *EF* at right angles in G.**

**Suppose the problem solved, *def* being a segment similar to *DEF* and having its surface equal to that of *ABB';* and complete the figure for *def* as for *DEF,* corresponding points being denoted by small and capital letters respectively.**

**Join *AB, DF, df.***

**Now, since the surfaces of the segments *def, ABB'* are equal, so are the circles on *df, AB* as diameters;**



**[I. 42, 43]**

**that is, *df = AB.***

**From the similarity of the segments *DEF, def* we obtain *d'd : dg = D' D : DG,***

**and *dg : df=DG :DF;*whence *d'd : df =D'D : DF,***

***or d'd : AB =D'D : DF.*But *AB, D'D, DF* are all given;**

**therefore *d'd* is given. Accordingly the synthesis is as follows.**

**Take *d'd* such that**

***d'd : AB =D'D :DF.* (I)**

**Describe a circle on *d'd* as diameter, and conceive a sphere constructed of**

**which this circle is a great circle.**

**Divide *d'd* at *g* so that**

***d'g : gd=D'G : GD,***

**and draw through *g* a plane perpendicular to *d'd* cutting off the segment *def***

**of the sphere and intersecting the plane of the great circle in *ef.* The segments**

***def, DEF* are thus similar, and**

***dg : df =DG : DF.***

**But from above, *componendo,***

***d'd : dg= D'D : DG.***

**Therefore, *ex aequali, d'd : df =D'D : DF,***

**whence, by (1), *df =AB.***

**Therefore the segment *def has* its surface equal to the surface of the segment**

***ABB' [I.* 42, 43], while it is also similar to the segment *DEF.***

**PROPOSITION *7 (PnosLEm)***

***From a given sphere to cut off* a *segment by a plane so that the segment may have a given ratio to the cone which has the same base as the segment and equal height.***

**Let *AA'* be the diameter of a great circle of the sphere. It is required to draw a plane at right angles to *AA!* cutting off a segment, as *ABB',* such that the segment *ABB'* has to the cone *ABB'* a given ratio.**

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***Analysis.***

**Suppose the problem solved, and let the plane of section cut the plane of the**

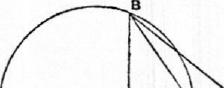
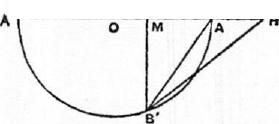
**great circle in *BB',* and the diameter *AA'* in *M.* Let 0 be the centre of the**

**sphere.**

**Produce *OA* to *H* so that**

***OA'-FA'M : A'M=HM : MA.* (1)**

**Thus the cone *HBB'* is equal**



o **to the segment *ABB'.***

**[Prop. 2]**

**Therefore the given ratio**

c **must be equal to the ratio of  
the cone *HBB'* to the cone *ABB', i.e.* to the ratio *HM : MA.***

a **Hence the ratio *OA' +  
A'M : A'M* is given; and**

**therefore *A'M* is given. *Scopcoµbs.***

**Now *OA' :A'M>OA' : A'A,***

**so that *OA'-FA'M :21`111>OA'-l-A'A : A'A***

**>3 : 2.**

**Thus, *in order that a solution may be possible, it is a necessary condition that***

***the given ratio must be greater than* 3 : 2.**

**The *synthesis* proceeds thus.**

**Let *AA'* be a diameter of a great circle of the sphere, *0* the centre.**

**Take a line *DE,* and a point *F* on it, such that *DE : EF* is equal to the given**

**ratio, being greater than 3 : 2.**

**Now, since *OA'+A'A : A'A* =3 : 2,**

***DE :EF>OA'-FA'A :A'A,***

**so that *DF : FE>OA' : A'A.*Hence a point *M* can be found on *AA'* such that**

***DF : FE = OA' : A'M.* (2)**

**Through *M* draw a plane at right angles to *AA'* intersecting the plane of the**

**great circle in *BB',* and cutting off from the sphere the segment *ABB'.***

**As before, take *H* on *OA* produced such that**

***: A'M=HM : MA.***

**Therefore *HM : MA =DE : EF,* by means of (2).**

**It follows that the cone *HBB',* or the segment *ABB', is* to the cone *ABB'* in**

**the given ratio *DE : EF.***

**PROPOSITION 8**

***If a sphere be cut by* a plane *not passing through the centre into two segments***

***A'BB', ABB', of which A'BB' is the greater, then the ratio***

***(segmt. A'BB') : (segmt. ABB')***

***<(surface of A'BB')2 : (surface of ABB')2***

***but> (surface of A'BB'); : (surface of ABB')'.***

**Let the plane of section cut a great circle *A'BAB'* at right angles in *BB',* and**

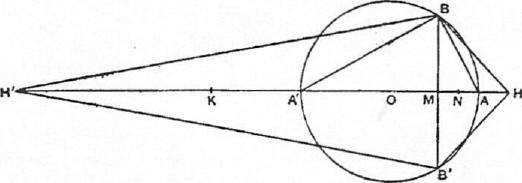
**let *AA'* be the diameter bisecting *BB'* at right angles in *M.***

**Let *0* be the centre of the sphere.**

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**Join *A'B, AB.***

**As usual, take *H* on *OA* produced, and *H'* on *OA'* produced, fig that**



***OA'+A'M : A'M =HM :MA, (1)***

***OA- AM : AM = H'M : MA',* (2)**

**and conceive cones drawn each with the same base as the two segments and with apices *H, H'* respectively. The cones are then respectively equal to the segments [Prop. 2], and they are in the ratio of their heights *HM, H'M.* Also**

**(surface of *A'BB') :* (surface of *ABB')=A'B2* : *AB2* [I. 42, 43]**

***=A'M : AM.***

**We have therefore to prove**

1. **that *H'M : MH<A1M2 : MA2,***
2. **that *H'M : MH>A'Ml : MAI.***
3. **From (2) above,**

***A'M : AM = H'M :04-1-AM***

***=H'A' : OA',* since *()A =0A'.***

**Since *A'M> AM, H'A' >OA' ;* therefore, if we take *K* on *H'A'* so that**

***OA' =41K, K* will fall between *H'* and *A'.***

**And, by (1), *A'M : AM—gm*** *;wt.*

**Thus *KM : MH=H'A' : AR,* singe *A'K =OA',***

***>H'M ;MK.***

**Therefore *411-1 <KM* 2.**

**It follows that**

***H'M •MH : MI-12<KM'***

**or *: MH <KM* 2 : *MR* 2**

**<44 W2 : *A* M2, by (1).**

1. ***Since QA1=0A,  
   A'M .MA < A/0 • OA,***

**or *A'M :OA' <OA : AM***

***<H'A' : A'M,* by means of (2).**

**Therefore *A'M* 2 < *H'4' • 0441.'***

***<11'A' .A'K.***

**Take a point *N* on *A'A* such that**

***A'N2.1i'A'* • *A'K.***

**Thus *H'A' : A/K=A/N2 : A'K'.*** (3)

**Also *: 41'N =41N : A'K,***

**and,** *componendo,*

***H'N : A'N =NK : A'K,***

ON TILE SPHERE AND CYLINDER II 43

whence A'N2 : A'K2 =1/W2 : *NK2.*

Therefore, by (3),

*H'A' : A'K = H'N2 : NK2.*

*Now H'M : MK>H'N : NK.*

Therefore *H'M2* : *MK2>H'A' : A'K*

*> H'A' : OA'*

*>A'M : MA,* by (2), as above,

*>0A'-FA'Al :A1H,* by (1),

*> KM : MH.*

Hence ***H'M2* : *MH2* =** *(H'M2 : MK2) • (KM2 : MH2)*

*> (KM : MH) • (KM2 : MH2).*

It follows that

*H'M* : *MH>KAla*

*>A' : AMT,* by (1).

**PROPOSITION** 9

*Of all segments of spheres which have equal surfaces the hemisphere is the greatest in volume.*

Let *ABA'B'* be a great circle of a sphere, *AA'* being a diameter, and *0* the centre. Let the sphere be cut by a plane, not passing through *0,* perpendicular to *AA' (at M),* and intersecting the plane of the great circle in *BB'.* The seg­ment *ABB'* may then be either less than a hemisphere as in Fig. 1, or greater than a hemisphere as in Fig. 2.

Let *DED'E'* be a great circle of another sphere, *DD'* being a diameter and *C* the centre. Let the sphere be cut by a plane through *C* perpendicular to *DD'* and intersecting the plane of the great circle in the diameter *EE'.*

Suppose the surfaces of the segment *ABB'* and of the hemisphere *DEE'* to be equal.

|  |  |
| --- | --- |
| Since the surfaces are equal, *AB =DE.*  **Fig. 3.**  **Fig. 1.**  Now, in Fig. *1, AB2>2AM2* and <2A02,  and, in Fig. 2, *AB' <2AM2* and >2A02. Hence, if *R* be taken on *AA'* such that  *AR2=iAB2,* | CI. 42, 43] |

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***R* will fall between *0* and *M.***

**Also, since *AB2 = DE', AR= CD.***

**Produce *OA'* to *K* so that *OA' = A'K,* and produce *A'A* to** *H* so that

***A'K : A'M =HA : AM,***

**or, *componendo, A'K-FA'M : A'M =HAI : MA.*** *(1)*

**Thus the cone *HBB'* is equal to the segment *ABB'.*** [Prop. 2]  
**Again, produce *CD* to *F* so that *CD = DF,* and the cone *FEE'*** will be equal

**to the hemisphere** *DEE'.* [Prop. 2]

**Now *AR •RA' > AM •MA',***

**and *AR2=1.4B2=iAM •AA' =AM •A'K.***

**Hence**

***AR •RA'-FRA2>AM•MA1-FAM •A'K,***

**or *AA' •AR> AM .MK***

***>HM •A'M,* by (1).**

Therefore ***AA' : A'M>HM : AR,***

or *AB2* : ***BM2>HM : AR,***

***i.e.*** *AR2* : *BM2>HM* : ***2AR,* since *AB2=2AR2,***

***>HM :CF.***

**Thus, since *AR= CD,* or *CE,***

**(circle on diam. *EE') :* (circle on diam.** *BB') > HM :*

**It follows that**

**(the cone *FEE')>* (the** cone *HBB' ),*

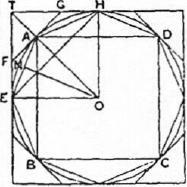
**and therefore** the **hemisphere *DEE'* is** greater in volume than the segment

***ABB'.***

MEASUREMENT OF A CIRCLE

PROPOSITION 1

*The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.*



K

Let *ABCD* be the given circle, *K* the triangle described.

Then, if the circle is not equal to *K,* it must be either greater or less.

1. If possible, let the circle be greater than *K.*

Inscribe a square *ABC!),* bisect the arcs *AB, BC, CD, DA,* then bisect (if necessary) the halves, and so on, until the sides of the inscribed polygon whose angular points arc the points of division subtend segments whose sum is less than the excess of the area of the circle over *K.*

Thus the area of the polygon is greater than *K.*

Let *AE* be any side of it, and ON the perpendicular on *AB* from the centre *0.*

Then ON is less than the radius of the circle and therefore less than one of the sides about the right angle in *K.* Also the perimeter of the polygon is less than the circumference of the circle, i.e. less than the other side about the right angle in *K.*

Therefore the area of the polygon is less than *K;* which is inconsistent with the hypothesis.

Thus the area of the circle is not greater than *K.*

1. If possible, let the circle be less than *K.*

Circumscribe a square, and let two adjacent sides, touching the circle in *E,*

*H,* meet in *T.* Bisect the arcs between adjacent points of contact and draw the

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tangents at the points of bisection. Let *A* be the middle point of the arc *Ell,*

and *FAG* the tangent at *A.*

Then the angle *TAG* is a right angle.

Therefore *TG> GA*

*>GH.*

It follows that the triangle *FTC:* is greater than half the area *TEAH.* Similarly, if the arc *AH* be bisected and the tangent at the point of bisection be drawn, it will cut off from the area *GAH* more than one-half.

Thus, by continuing the process, we shall ultimately arrive at a circum­scribed polygon such that the spaces intercepted between it and the circle are together less than the excess of *K* over the area of the circle.

Thus the area of the polygon will be less than *K.*

Now, since the perpendicular from *0* on any side of the polygon is equal to the radius of the circle, while the perimeter of the polygon is greater than the circumference of the circle, it follows that the area of the polygon is greater than the triangle *K;* which is impossible.

Therefore the area of the circle is not less than *K.*

Since then the area of the circle is neither greater nor less than *K,* it is equal to it.

PROPOSITION 2

*The area of a circle is to the square on its diameter as 11 to* 14'.

PROPOSITION 3

*The ratio of the circumference of any circle to its diameter is less than* 3+ *but greater than 3.'4'.*

I. Let *AB* be the diameter of any circle, 0 its centre, A*C* the tangent at A and let the angle *AOC* be one-third of a right angle.

Then *OA : AC[= N/3 :* 1]> 265 : 153, (11

and *OC* : *CA[=* 2 : 1] =300: 153. (21

*First,* draw *OD* bisecting the angle *AOC* and meeting *AC* in *1).*

Now CO : *OA =CD : DA,* [Eucl. VI.

so that *[C0+0A : OA =CA :DA,* or]

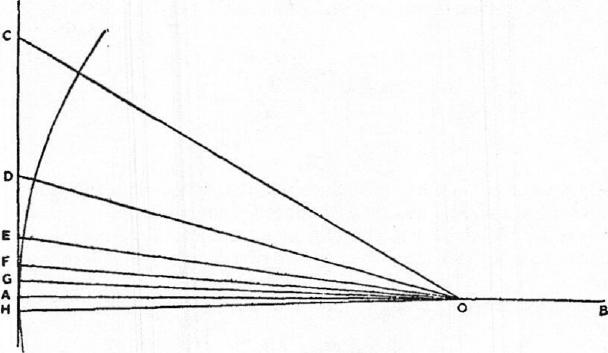
*CO-FOA :CA =OA :AD.*Therefore [by (1) and (2)]

*OA : AD>571 :* 153. (3)

'The text of this proposition is not satisfactory, and Archimedes cannot have placed it before Proposition 3, as the approximation depends upon the result of that proposition.

'In view of the interesting questions arising out of the arithmetical content of this proposi­tion of Archimedes, it is necessary, in reproducing it, to distinguish carefully the actual steps set out in the text as we have it from the intermediate steps (mostly supplied by Eutocius) which it is convenient to put in for the purpose of making the proof easier to follow. Accord­ingly all the steps not actually appearing in the text have been enclosed in square brackets, in order that it may be clearly seen how far Archimedes omits actual calculations and only gives results. It will be observed that he gives two fractional approximations to ,./3 (one being less and the other greater than the real value) without any explanation as to how he arrived at them; and in like manner approximations to the square roots of several large numbers which arc not complete squares are merely stated.

|  |  |  |
| --- | --- | --- |
| **Hence**  **BO that** | **MEASUREMENT OF A CIRCLE**  ***OD2 AD2[=(0A2-1-AD!) : AD2***  ***>* (5712+ 1532) : 1532]**  **>349450 : 23409,**  ***OD :DA* >591i : 153.** | **47**  **(4)** |



*Secondly,* **let *OE* bisect the angle A OD, meeting *AD* in *E.***

**[Then *DO : OA = DE : EA,***

**so that *DO+OA : DA =OA : AE.]***

**Therefore *OA : AE [>* (5911+571) 153, by (3) and (4)]**

**> 11621 : 153. (5)**

**[It follows that**

***0E2 : EA2> 1(1162)02+1531* : 1532**

**>(1350534-N+23409) :23409**

**>1373943U : 23409.1**

**Thus *OE : EA > 11721 :* 153. (6)**

***Thirdly, let OF* bisect the angle *AOE* and meet AE in *F.***

**We thus obtain the result [corresponding to (3) and (5) above] that**

***OA : AF [>* (11621+11721) :153]**

**>23341 : 153. (7)**

**[Therefore *OFZ* : *FA2>* [ (23341)2+1532} : 1532**

**>5472132A : 23109.]**

**Thus *OF : FA > 23391* : 153. (8)**

***Fourthly,* let *OG* bisect the angle .40F, meeting *A* in *G.***

**We have then**

**OA : AG [> (23341+23391) : 153, by means of (7) and (8)]**

**>46731 : 153.**

**Now the angle *AOC,* which is one-third of a right angle, has been bisected**

**four times, and it follows that**

**Z *AOG= (a* right angle).**

**Make the angle *AOH* on the other side of *OA* equal to the angle *AOG,* and**

**let GA produced meet 0/1 in *11.***

**Then L GO//:-..-.211- (a right angle).**

*4-8* ARCHINIEDES

Thus *GH* is one side of a regular polygon of 96 sides circumscribed to the given circle.

And, since *OA :* AG>46731 : 153,

while *A B =20A, GH =2AG,*

it follows that

*AB :* (perimeter of polygon of 96 sides)[> 46731 : 153 X96]

>4673₹ : 14688.

14688-3+ 6672

But

46734 467n

[

6671   
<3+4672i

<34-.

Therefore the circumference of the circle (being less than the perimeter of the polygon) is a *fortiori* less than 3+ times the diameter *AB.*

*II.* Next let *AB* be the diameter of a circle, and let *AC,* meeting the circle in *C,* make the angle *CAB* equal to one-third of a right angle. Join *BC.*

Then *AC : CB[= :1]<1351* 780.

*First,* let *AD* bisect the angle *BAC* and meet *BC in d* and the circle in *D.*

Join *BD.*

Then L *BAD= ZdAC*

*= L dBD,*

and the angles at D, *C* are both right angles.

It follows that the triangles *ADB,[ACd], BDd* are similar.

Therefore *AD : DB = BD : Dd*

*[=AC :Cd]*

*=AB : Bd* [Eucl. vi. 3]

*=AB-FAC : Bd+Cd*

*=AB-E-4C : BC*

or *BA -FAC : BC =AD :DB.*

[But *AC : CB <1351* : 780, from above,

while *BA : BC =* 2 : 1

=1560 : 780.]

Therefore *AD : DB* <2911 : 780. (l)

[Hence AB2 BD2<(29112+7802) : 780'

<9082321 : 608400.]

Thus *AB :* BD<30131 : 780. (2)

*Secondly,* let *AE* bisect the angle *BAD,* meeting the circle in *E;* and let *BE*

be joined.

Then we prove, in the same way as before, that

*AE :EB[=BA+AD :BD*

<(30134+2911) : 780, by (1) and (2)]

<59241 : 780

<5924ixilg : 780XA

<1823 : 240. (3)

[Hence *AB2 BE2 <* (18232+2402) : 2402

<3380929 : 57600.]

Therefore *AB : BE <1838A :* 240. (4)  
*Thirdly,* let *AF* bisect the angle *BAE,* meeting the circle in *F.*

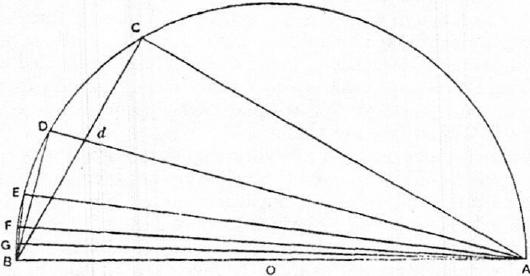
MEASUREMENT OF A CIRCLE 49

Thus *AF :FBC=BA+AE : BE*

*<366119r :* 240, by (3) and (4)]

<3661AX-14 : 240X -H

<1007 : 66. (5)



[It follows that

*AB2 : BF' <* (10072+66') : 662

<1018405 : 4356]

Therefore *AB : BF <1009i* : 66. (6)  
*Fourthly,* let the angle *BAF* be bisected by *AG* meeting the circle in *G.*

Then *AG :GB[=BA+AF : BF]*

<2014 : 66, by (5) and (6).

[And *AB2* : *BG'<* 1(2014)2+661 : 662

<4069284k : 4356.]

Therefore *AB : BG* <20171 : 66,

whence *BG : AB>* 66 : 20171. (7)

[Now the angle *BAG* which is the result of the fourth bisection of the angle

*BAC,* or of one-third of a right angle, is equal to one-fortyeighth of a right

angle.

Thus the angle subtended by *BG* at the centre is

(a right angle).]

Therefore *BG* is a side of a regular inscribed polygon of 96 sides.

It follows from (7) that

(perimeter of polygon) : AB [> 96 X66 : 20171]

>6336 : 20174.

6336

And

20171>31t.

Much more then is the circumference of the circle greater than 344 times the

diameter.

Thus the ratio of the circumference to the diameter

<3+ but >344.

ON CONOIDS AND SPHEROIDS

INTRODUCTION'

**"ARCHIMEDES** to Dositheus greeting.

"In this book I have set forth and send you the proofs of the remaining t heorems not included in what I sent you before, and also of some others dis­covered later which, though I had often tried to investigate them previously, I had failed to arrive at because I found their discovery attended with some difficulty. And this is why even the propositions themselves were not published with the rest. But afterwards, when I had studied them with greater care, I discovered what I had failed in before.

"Now the remainder of the earlier theorems were propositions concerning the right-angled conoid [paraboloid of revolution]; but the discoveries which I have now added relate to an obtuse-angled conoid [hyperboloid of revolution] and to spheroidal figures, some of which I call *oblong* and others *flat."*

I. "Concerning the *right-angled conoid* it was laid down that, if a section of a right-angled cone [a parabola] be made to revolve about the diameter [axis] which remains fixed and return to the position from which it started, the figure comprehended by the section of the right-angled cone is called a *right-angled conoid,* and the diameter which has remained fixed is called its *axis,* while its *vertex* is the point in which the axis meets the surface of the conoid. And if a plane touch the right-angled conoid, and another plane drawn parallel to the tangent plane cut off a segment of the conoid, the base of the segment cut off is defined as the portion intercepted by the section of the conoid on the cutting plane, the *vertex* [of the segment] as the point in which the first plane touches the conoid, and the *axis* [of the segment] as the portion cut off within the seg­ment from the line drawn through the vertex of the segment parallel to the axis of the conoid.

"The questions propounded for consideration were"

1. "why, if a segment of the right-angled conoid be cut off by a plane at right angles to the axis, will the segment so cut off be half as large again as the cone which has the same base as the segment and the same axis, and"
2. "why, if two segments be cut off from the right-angled conoid by planes drawn in any manner, will the segments so cut off have to one another the duplicate ratio of their axes."

II. "Respecting the *obtuse-angled conoid* we lay down the following prem-isses. If there be in a plane a section of an obtuse-angled cone [a hyperbola], its

**'The** whole of this introductory matter, including the definitions, is translated literally from the Greek text in order that the terminology of Archimedes may be faithfully repre­sented. When this has once been set out, nothing will be lost by returning to modern phrase­ology and notation. These will accordingly be employed, as usual, when we come to the actual propositions of the treatise.

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diameter [axis], and the nearest lines to the section of the obtuse-angled cone

*[i.e.* the asymptotes of the hyperbola], and if, the diameter [axis] remaining fixed, the plane containing the aforesaid lines be made to revolve about it and return to the position from which it started, the nearest lines to the section of the obtuse-angled cone [the asymptotes] will clearly comprehend an isosceles cone whose vertex will be the point of concourse of the nearest lines and whose axis will be the diameter [axis] which has remained fixed. The figure compre­hended by the section of the obtuse-angled cone is called an *obtuse-angled conoid* [hyperboloid of revolution], its *axis* is the diameter which has remained fixed, and its *vertex* the point in which the axis meets the surface of the conoid. The cone comprehended by the nearest lines to the section of the obtuse-angled cone is called [the cone] *enveloping the conoid,* and the straight line between the vertex of the conoid and the vertex of the cone enveloping the conoid is called [the line] *adjacent to the axis.* And if a plane touch the obtuse-angled conoid, and another plane drawn parallel to the tangent plane cut off a segment of the conoid, the *base* of the segment so cut off is defined as the por­tion intercepted by the section of the conoid on the cutting plane, the *vertex* [of the segment] as the point of contact of the plane which touches the conoid, the *axis* [of the segment] as the portion cut off within the segment from the line drawn through the vertex of the segment and the vertex of the cone enveloping the conoid; and the straight line between the said vertices is called *adjacent to the axis.*

"Right-angled conoids are all similar; but of obtuse-angled conoids let those be called similar in which the cones enveloping the conoids are similar.

"The following questions are propounded for consideration":

1. "why, if a segment be cut off from the obtuse-angled conoid by a plane at right angles to the axis, the segment so cut off has to the cone which has the same base as the segment and the same axis the ratio which the line equal to the sum of the axis of the segment and three times the line adjacent to the axis bears to the line equal to the sum of the axis of the segment and twice the line adjacent to the axis, and"
2. "why, if a segment of the obtuse-angled conoid be cut off by a plane not at right angles to the axis, the segment so cut off will bear to the figure which has the same base as the segment and the same axis, being a segment of a cone, the ratio which the line equal to the sum of the axis of the segment and three times the line adjacent to the axis bears to the line equal to the sum of the axis of the segment and twice the line adjacent to the axis."

III: "Concerning spheroidal figures we lay down the following premisses. If a section of an acute-angled cone [ellipse] be made to revolve about the greater diameter [major axis] which remains fixed and return to the position from which it started, the figure comprehended by the section of the acute-angled cone is called an *oblong spheroid.* But if the section of the acute-angled cone revolve about the lesser diameter [minor axis] which remains fixed and return to the position from which it started, the figure comprehended by the section of the acute-angled cone is called a *flat spheroid.* In either of the spheroids the

*axis isdefined* as the diameter [axis] which has remained fixed, the *vertex* as the point *i*n which the axis meets the surface of the spheroid, the *centre* as the

middle point of the axis, and the *diameter* as the line drawn through the centre at right angles to the axis. And, if parallel planes touch, without cutting, either

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of the spheroidal figures, and if another plane be drawn parallel to the tangent planes and cutting the spheroid, the *base* of the resulting segments is defined as the portion intercepted by the section of the spheroid on the cutting plane,

their *vertices* as the points in which the parallel planes touch the spheroid, and

their *axes* as the portions cut off within the segments from the straight line joining their vertices. And that the planes touching the spheroid meet its sur­face at one point only, and that the straight line joining the points of contact passes through the centre of the spheroid, we shall prove. Those spheroidal figures are called *similar* in which the axes have the same ratio to the 'diam­eters.' And let segments of spheroidal figures and conoids be called *similar* if they are cut off from similar figures and have their bases similar, while their axes, being either at right angles to the planes of the bases or making equal angles with the corresponding diameters [axes] of the bases, have the same ratio to one another as the corresponding diameters [axes] of the bases.

"The following questions about spheroids are propounded for considera­tion,"

1. "why, if one of the spheroidal figures be cut by a plane through the centre at right angles to the axis, each of the resulting segments will be double of the cone having the same base as the segment and the same axis; while, if the plane of section be at right angles to the axis without passing through the centre, (a) the greater of the resulting segments will bear to the cone which has the same base as the segment and the same axis the ratio which the line equal to the sum of half the straight line which is the axis of the spheroid and the axis of the lesser segment bears to the axis of the lesser segment, and *(b)* the lesser segment bears to the cone which has the same base as the segment and the same axis the ratio which the line equal to the sum of half the straight line which is the axis of the spheroid and the axis of the greater segment bears to the axis of the greater segment";
2. "why, if one of the spheroids be cut by **a** plane passing through the centre but not at right angles to the axis, each of the resulting segments will be double of the figure having the same base as the segment and the same axis and consisting of a segment of **a** cone.
3. "But, if the plane cutting the spheroid be neither through the centre nor at right angles to the axis, (a) the greater of the resulting segments will have to the figure which has the same base as the segment and the same axis the ratio which the line equal to the sum of half the line joining the vertices of the seg­ments and the axis of the lesser segment bears to the axis of the lesser segment, and (b) the lesser segment will have to the figure with the same base as the segment and the same axis the ratio which the line equal to the sum of half the line joining the vertices of the segments and the axis of the greater segment bears to the axis of the greater segment. And the figure referred to is in these cases also a segment of a cone.

"When the aforesaid theorems are proved, there are discovered by means of them many theorems and problems.

"Such, for example, are the theorems":

1. "that similar spheroids and similar segments both of spheroidal figures and conoids have to one another the triplicate ratio of their axes, and"
2. "that in equal spheroidal figures the squares on the `diameters' are re­ciprocally proportional to the axes, and, if in spheroidal figures the squares on

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the `diameters' are reciprocally proportional to the axes, the spheroids are equal.

"Such also is the problem, From a given spheroidal figure or conoid to cut off a segment by a plane drawn parallel to a given plane so that the segment cut off is equal to a given cone or cylinder or to a given sphere.

"After prefixing therefore the theorems and directions which are necessary for the proof of them, I will then proceed to expound the propositions them­selves to you. Farewell."

DEFINITIONS

"If a cone be cut by a plane meeting all the sides [generators] of the cone, the section will be either a circle or a section of an acute-angled cone [an ellipse]. If then the section be a circle, it is clear that the segment cut off from the cone towards the same parts as the vertex of the cone will be a cone. But, if the section be a section of an acute-angled cone [an ellipse], let the figure cut off from the cone towards the same parts as the vertex of the cone be called a *segment of a cone.* Let the *base* of the segment be defined as the plane compre­hended by the section of the acute-angled cone, its *vertex* as the point which is also the vertex of the cone, and its *axis* as the straight line joining the vertex of the cone to the centre of the section of the acute-angled cone.

"And if a cylinder be cut by two parallel planes meeting all the sides [gen­erators] of the cylinder, the sections will be either circles or sections of acute-angled cones [ellipses] equal and similar to one another. If then the sections be circles, it is clear that the figure cut off from the cylinder between the parallel planes will be a cylinder. But, if the sections be sections of acute-angled cones [ellipses], let the figure cut off from the cylinder between the parallel planes be called a *frustum of a cylinder.* And let the *bases* of the frustum be defined as the planes comprehended by the sections of the acute-angled cones [ellipses], and the *axis* as the straight line joining the centres of the sections of the acute-angled cones, so that the axis will be in the same straight line with the axis of the cylinder."

**LEMMA**

*If in an ascending arithmetical progression consisting of the magnitudes* **A1,**

**A2, • • 'An** *the common difference be equal to the least term A1, then*

*n* -An <2(ArfA2-1-• • • •±An),

*and >2(Ai-F A 2+ • • •+An-1).*

[The proof of this is given incidentally in the treatise *On Spirals,* Prop. 11. By placing lines side by side to represent the terms of the progression and then producing each so as to make it equal to the greatest term, Archimedes gives the equivalent of the following proof.

If Sn=Al+A2+ • • •-f-An-1-1-An,

we have also Sn=An+An-i+An-2+ • •

And Ai+A....4=A2+An-2=-- • • • =An.

Therefore 2Sn= (n-1-1)An,

whence n *•An<28„,*

and n *•An>2S.-i.*

Thus, if the progression is a, 2a, • • •na,

0 n(n+1)

**On—** 2 a,

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and n2a<2S,„

but >2Sn-1.]

**PROPOSITION I**

*If A1, B1, C1, • • •K1* and *A2,* **B2, C2, • • *•K2*** *be two series of magnitudes such that*

*AI:Bi=A2:* **B2, (°)**

*B1* **: CI =B2 : C2,** *and so on,*

*and if* **A3, B3, C3, • • •K3** *and A1, B4,* C4, • • •K4 *be two other series such that*

*AI: A3= A2 :* **A4, 1 (3)**B1 **: B3= B2 :** *B4,* and *so on,* j

*then (Ai+Bi+CI-1-• • • .+Ki) :(A3-1-1334-Ca+ • • .+K3)*

*=(A2-1-B2-FC2+ • • •+K2) :(A.4-1-B4-1- • •* •+K4). The proof is as follows.

Since **A3 :A/=A4 : A2,**

and AI :B1=A2 :B2

while B1 : **B3 = B2 :** *B4,*

we have, *ex aequali,* **A3 : B3=** *A4 :* **B4.** (7)

Similarly **B3 : C3 =B4 : C4,** and so on.

Again, it follows from equations **(a)** that

441:A2=131:B2=GL : C2= **• • •**

Therefore

Al : **A2= (Al+Bl+Cl+ •+Kl) (212+B2+ •** .+K2),

or (A1-1-.81-1-C1-1- • • •-FKI) : Ai= (A2-1-B2+C2-1- • • *•+K2)* : *A* 2;

and Ai *:A3=A2:A4,*

while from equations (7) it follows in like manner that

A3 : (A3+B3-FC3+ • • •+K3) =A4 : (A4+1344-C4+ • • •A-K4).

By the last three equations, *ex aequali,*

(A1+B1+C1+ *- • •+I(1) :(A3+B3i-C3+ • • •+K3)*

*=(A24-B2i-C2+ • "-FK2) :(A4-1-B4-4-C44- • •* **•4-K4)•**

**COR.** If any terms in the third and fourth series corresponding to terms in

the first and second be left out, the result is the same. For example, if the last

terms **K3, K4** are absent,

(A1+B1+C1+ • • •+Ki) : (A3-f-B3+C3+ • • •+./3)

= (A2+B2-+-C24- • • --FK2) : (A4-+-B44-C4-1-- • • •.4-I41,

where *I* immediately precedes *K* in each series.

**LEMMA TO PROPOSITION** 2  
*[On Spirals,* Prop. 10.]

*If A1, A* 2, *A3, • • • A n be n lines forming an ascending arithmetical progression in*

*which the common difference is equal to the least term A1, then*

*(n+1)A„2-FAi(Ai+A2+A3+ • • •* +An) **=3(Al2+A22+A32+ • • •+An2).**

Let the lines A., An\_1, An-2, ...Ai be placed in a row from left to right.

Produce An...1, An\_2, ...AI until they are each equal to *An,* so that the parts

produced are respectively equal to A1, *A2....*.n-i.

Taking each line successively, we have

2A **n2 = 2An2,**

**(Zii-FAn-1)2=A/2+A2n-1+2A1 \*An-1,**

**(112+A n-2)2= *A* 22+ *A* 2n-2+2A 2 'An--2,**

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* A1.  
  And, by addition,

(n+ 1)21,,2 = 2(Ai2 +A 22+ • • • -1-21„2)

/•" -N

-1-2A1 .A.,-1+2A2 •A,,-.2-1- • • • -1--2An-1 •A1. Therefore, in order to obtain the re­quired result, we have to prove that 2(A1 •Ah\_H-A2 v1,2+ • • •-f-An\_i •A1)+ Al(441-i-A2-1--A3+ • • *•-E-An)*

*=Al2-1-A22+ - • •+A„2.* (a)  
Now

2A2.An-2-----AI-4A,,t, because *A2=2,41, 2A* **3 •A\*-3 =** *Al* •6A,3, because *A8=* 3A 1,

**A..3An-2 A14.1**

**Al A:**

**An, Aft\_2**

*2A„.\_1 •A =* Ai .2(n-1)A1.

It follows that

2(A1 •An\_i+A2 •An\_2+ • • • A-An-i •A1)+

**A3 A3 Al A1(A1+A2+ • • •—FA„)** *= A1* { A n+ 3A „\_.1+

5A „...t+ • • • + (2n — 1)A

And this last expression can be proved to be equal to

2112-FA22+ • • •+A„2.

For An2--=-Ai(n •An)

*=Ai{An+(n-1)An}*

=1411A,,-F2(A,,\_14-A2+ • • •-f•Ai)),

because (n —1 *)A„.*

+An-2+A2

Similarly A2,;\_i=i11{An\_I-1-2(A„\_2-FAn\_3+ • • •-f•Ai)},

At2=Ai(A2+2211),

Ai' •A1;

whence, by addition,

Al2-FA22-FA32+ • • •+44,2=AdAn+3A7,-1-1-5A.-2-1- • • •+(2n—I)Ail• Thus the equation marked (a) above **is** true; and it follows that

(n-1-1)A„2-FAI(A1-1-A2-i-A3-1- • • • *-FA.)=3(Al2-1-A22-1- • • •+An2).* **COR.** *1. From this it is evident that*

n ***•An'<3(Al2+A22-1- • •*** *•-•f-An2). (1)*

Also An2=At{A,.+2(An-i+An-2+ • • H-A1)}, as above.

so that An2>A1(An+A.-1+ • • •-kAi),

and therefore

* A,,2+.111(Ari-A24- • • *• FA.) <2A,,2 .  
  It follows from the proposition that*

*n•An2>3(Al2-i-A224- • •* •-f-A2.-1). (2)  
**COR.** 2. All these results will hold if we substitute *similar figures* for squares on all the lines; for similar figures are in the duplicate ratio of their sides.

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**PROPOSITION** 2

*If Ai,* **A2 • •** *•A T, be any number of areas such that*

*Al=ax+x2,*

*A2=a .2x+(2x)2,  
As= a .3x+ (3x)2 ,*

*A n= a •ns-1-(nx)2,*

*then n .A„ : (Ai+ A2+ • • •- A.) <(a+nx) (c1-1-11-D •*

*and n•An:(Ai+A2+ • • .+A,1)>(a+nx) 6+7).*

For, by the Lemma immediately preceding Prop. 1,

n *•anx<(ax+a .2x+ • • • +a .nx),*

and >2(ax-1-a .2x-1- • • • +a •n—*lx).* by the Lemma preceding this proposition,

*n .(n:c)2* <3 { x2+ (2x)2+(3x)2+ • • • ± *(nx)21*

and > 3{ x2-1- (2:02+ • • • (n — 1 x)2}

Hence

*an*2*2x n(nx)2*

*<[(ax+x2)+* fa .2x+ (2x)21 + • • • + *{a* **•itx-f-** *(nx)2}j,*

3

and

*>[(ax+ x2) +{*a -2x+ (2x)2} + • • • ± { a •n— *lx (n — lx)2}],*

or 2 3

*an2x n(nx)2* **<A14-A2+ • • •** +A.,

and >A1-1-A2+ • • •+A.--i.

It follows that

*n •A„ : (Al+ A2+ • • •* +A.)<n{a •nx+ *(nx)2} Icc+11(7 31.42* },

or *n* •A„ : (A1-1-A2+ • • *•-f-An)<(a-Enx) :* (2+3-:-T);

also *n.A„: (A1+A2+ • • • +An-i)> (a+nx) :* **6+1f).**

**PROPOSITION** 3

(1) *If TP, TP' be two tangents to any conic meeting in T, and if Qq, Q'q' be* any

*two chords parallel respectively to TP, TP' and meeting in 0, then*

*QO .0q : Q/0 •Oq' = TP2 : TP".*

"And this is proved in the elements of conics."'

(2) *If QQ' be a chord of a parabola bisected in V by the diameter PV, and if P*

*be of constant length, then the areas of the triangle PQQ' and of the segment PQQ'*

*are both constant whatever be the direction of QQ'.*

Let *ABB'* be the particular segment of the parabola whose vertex is A, so

that *BB' is* bisected perpendicularly by the axis at the point *H,* where AH *= PV.*

Draw *QD* perpendicular to PV.

**1In** the treatises on conics by Aristaeus and Euclid.

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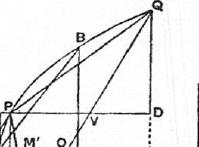
**Let pa be the parameter of the principal ordinates, and let p be another line**

**of such length that**

***QV'* : QD2=p :** *p.;*

**it will then follow that p is equal to the par-**

**ameter of the ordinates to the diameter *PV,***



**i.e. those which are parallel to *QV.***

**"For this is proved in the conics."'**

**Thus *QV2=p•PV.***

**And *BH2=* pa *-A H,* while *AH =PV.***

***Pap* Therefore *QV2* : *BH2 =* p : pa.**

**TA But QV2 : *QD2 =p* : pa;**

**hence *BH = QD.***



**Thus *BH•AH=QD •PV,***

**and therefore *LABB' = LPQQ';***

**that is, the area of the triangle *PQQ'* is con-**

**s, stant so long *as PV* is of constant length. Hence also the area of the segment *PQQ'* is constant under the same condi­tions; for the segment is equal to *4/PQQ'. [Quadrature of the Parabola,* Prop. 17 or 24.]**

**PROPOSITION *4***

***The area of any ellipse is to that of the auxiliary circle as the minor axis to the***

***major.***

**Let *AA'* be the major and *BB'* the minor axis of the ellipse, and let *BB'***

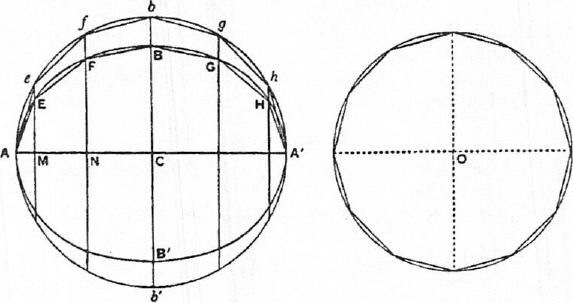
**meet the auxiliary circle in *b,* b'.**

**Suppose *0* to be such a circle that**

**(circle *AbA'b') : 0 = CA : CB.***

**Then shall *0* be equal to the area of the ellipse.**

**For, if not, *0* must be either greater or less than the ellipse.**



***b'***

f

b

**I. If possible, let *0* be greater than the ellipse.**

**We can then inscribe in the circle *0* an equilateral polygon of 4n sides such**

**that its area is greater than that of the ellipse. [cf. *On the Sphere and Cylinder,***

**I.6.]**

**'The theorem which is here assumed by Archimedes as known ... is easily deduced from**

**Apollonius I. 49....**

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**Let this be done, and inscribe in the auxiliary circle of the ellipse the polygon *AefbghA' .* similar to that inscribed in *0.* Let the perpendiculars eM, *IN,.* on *AA'* meet the ellipse in *E, F,...* respectively. Join *AE, EF, FB,....***

**Suppose that *P'* denotes the area of the polygon inscribed in the auxiliary circle, and *P* that of the polygon inscribed in the ellipse.**

**Then, since all the lines elf *fN,• • •* are cut in the same proportions at *E,***

***F, • • •***

**i.e. *eM : EM =fN FN = • • • =bC : BC,***

**the pairs of triangles, as *eAM, EAM,* and the pairs of trapeziums, as *eMNf,***

***EMNF,* are all in the same ratio to one another as *bC* to *BC,* or as *CA* to *CB.***

**Therefore, by addition,**

***P' :P=CA :CB.***

**Now *P' ;* (polygon inscribed in *0)***

**= (circle *AbA'6') :0***

***= CA : CB,* by hypothesis.**

**Therefore *P* is equal to the polygon inscribed in *0.***

**But this is impossible, because the latter polygon is by hypothesis greater**

**than the ellipse, and a *fortiori* greater than *P.***

**Hence *0* is not greater than the ellipse.**

**II. If possible, let 0 be less than the ellipse.**

**In this case we inscribe in the *ellipse* a polygon *P* with 4n equal sides such**

**that *P>0.***

**Let the perpendiculars from the angular points on the axis *AA'* be produced**

**to meet the auxiliary circle, and let the corresponding polygon *(P')* in the**

**circle be formed.**

**Inscribe in 0 a polygon similar to *P'.***

**Then *: P = CA : CB***

***=* (circle *AbA'b') :0,* by hypothesis,**

***=--P' :* (polygon inscribed in 0).**

**Therefore the polygon inscribed in 0 is equal to the polygon *P;* which is**

**impossible, because *P> O.***

**Hence 0, being neither greater nor less than the ellipse, is equal to it; and**

**the required result follows.**

**PROPOSITION 5**

***If AA', BB' be the major and minor axis of an ellipse respectively,* and *if d be the***

***diameter of* any *circle, then***

***(area of ellipse) : (area of circle) =AA' • BB' :***

**For**

**(area of ellipse) : (area of auxiliary circle) *=iBB' : AA'* [Prop. 4]**

***.AA' • BB' : AA'2.***

**And**

**(area of aux. circle) : (area of circle with diam. d) *=AA":* d2.**

**Therefore the required result follows *ex aequali.***

**PROPoSI'TION 6**

***The areas of ellipses are as the rectangles under their axes.***

**This follows at once from Props. *5.***

**Cox. *The areas of similar ellipses are as the squares of corresponding* oases,**

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**PROPOSITION** *7*

*Given an ellipse with centre C, and a line CO drawn perpendicular to its plane, it is possible to find a circular cone with vertex 0 and such that the given ellipse is a section of it[or, in other words, to find the circular sections of the cone with vertex* ***0*** *passing through the circumference of the ellipse].*

Conceive an ellipse with ***BB'*** as its minor axis and lying in a plane perpen­dicular to that of the paper. Let, *CO* be drawn perpendicular to the plane of the ellipse, and let *0* be the vertex of the required cone. Produce *OB, OC, OB',* and in the same plane with them draw *BED* meeting *OC, OR'* produced in *E, D* respectively and in such a direction that

*BE • ED : E02 =CA" :* CO2,

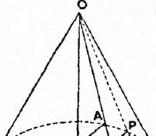
where *CA* is half the major axis of the ellipse.

"And this is possible, since

*BE • ED E02> BC • CB' :*CO2."

[Both the construction and this proposition are assumed as known.]

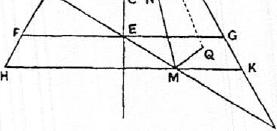
Now conceive a circle with *BD* as diam­eter lying in a plane at right angles to that of the paper, and describe a cone with this circle for its base and with vertex 0.



We have therefore to prove that the given ellipse is a section of the cone, or, if *P* be any point on the ellipse, that *P* lies

**8** on the surface of the cone.

Draw *PN* perpendicular to *BB'.* Join *ON* and produce it to meet *BD* in *M,* and let *MQ* be drawn in the plane of the circle on *BD* as diameter perpendicular to *BD* and meeting the circle in Q. Also let *FG, HK* be drawn through *E, M* respectively parallel to *BB'.*



We have then

*QM2:HM MK = BM • MD :HM • MK*

*=BE • ED : FE • EG*

*= (BE • ED : E02) • (E02 : FE • EC)*

*=(CA2 : CO2) •* (CO' : *BC • CB')*

*=CA' : CB'*

*=PN2 : BN • NB'.*

Therefore *QM2* : *PN2=HM •MK : BN • NB'*

*=0M2 :ON';*

whence, since *PN, QM* are parallel, *OPQ is a* straight line.

But *Q* is on the circumference of the circle on *BD* as diameter; therefore *OQ*

is a generator of the cone, and hence *P* lies on the cone.

Thus the cone passes through all points on the ellipse.

**PROPOSITION** 8

*Given an ellipse, a plane through one of its axes AA' and perpendicular to the plane of the ellipse, and a line CO drawn from C, the centre, in the given plane through AA' but not perpendicular to AA', it is possible to find a cone with vertex*

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*0 such that the given ellipse is a section of it [or, in other words, to find the circular sections of the cone with vertex 0 whose surface passes through the circumference of the ellipse].*

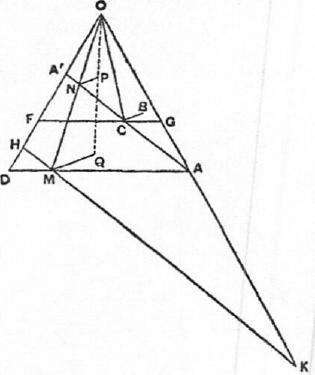
*By* hypothesis, *OA, OA'* are unequal. Produce ***OA'*** to ***D so*** that ***OA =OD.*** Join *AD,* and draw *FG* through *C* parallel to it.

The given ellipse is to be supposed to lie in a plane perpendicular **to the** plane of the paper. Let *BB'* be the other axis of the ellipse.

Conceive a plane through *AD* perpendicular to the plane of the paper, and in it describe either (a), if *CB2=FC • CG,* a circle with diameter *AD,* or *(b),* if not, an ellipse on *AD* as axis such that, if *d* be the other axis,

*d2 : AD2 =CB' : FC • CG.*

Take a cone with vertex 0 whose sur­face passes through the circle or ellipse just drawn. This is possible even when the curve is an ellipse, because the line from *0* to the middle point of *AD* is per­pendicular to the plane of the ellipse, and the construction is effected by means of Prop. 7.



Let *P* be any point on the given el­lipse, and we have only to prove that ***P*** lies on the surface of the cone so de­scribed.

Draw *PN* perpendicular to ***AA'.*** Join *ON,* and produce it to meet *AD* in *M.* Through *M* draw *HK* parallel to *A'A.*

Lastly, draw *MQ* perpendicular to the plane of the paper (and therefore perpendicular to both *HK* and ***AD)*** meeting the ellipse or circle about ***AD*** (and therefore the surface of the cone) in *Q.*

Then

*QM2* : ***HM • MK = (QM' :DM • MA) • (DM • MA :HM •*** *MK)*

*= (d2* : *AD2)* • *(FC • CG : A'C • CA) =(CB2 : FC • CG) • (FC • CG : A'C • CA) =CB2 :CA2*

**= *PN2* :** *A 'N • NA.*

Therefore, alternately,

*QM2* : *PN2=HM • MK :* ***A'N • NA***

***=0M2*** *:* ON2.

Thus, since *PN, QM* are parallel, *OPQ* is a straight line; and, Q being on the

surface of the cone, it follows that ***P*** is also on the surface of the cone.

Similarly all points on the ellipse are also on the cone, and the ellipse is

therefore a section of the cone.

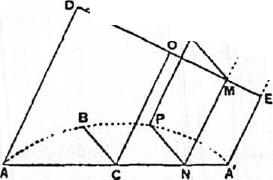
**PROPOSITION** 9

*Given an ellipse, a plane through one of its axes and perpendicular to that of the ellipse, and a straight line CO drawn from the centre C of the ellipse in the given plane through the axis but not perpendicular to that* axis, it *is possible to* find a

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*cylinder with axis OC such that the ellipse is* a *section of it [or, in other words, to find the circular sections of the cylinder with axis OC whose surface passes through the circumference of the given ellipse].*

Let *AA'* be an axis of the ellipse, and suppose the plane of the ellipse to be perpendicular to that of the paper, so that *OC* lies in the plane of the paper. Draw *AD, A'E* parallel to *CO,* and let *DE* be the line through *0* perpendicular



*.* to both *AD* and *A'E.  
......... • •*

* ...,E' We have now three different cases

,

according as the other axis *BB'* of the

* ellipse is (1) equal to, (2) greater than, •

**or** (3) less than, *DE.*

**(1)** Suppose *BB' = DE.*

Draw a plane through *DE* at right angles to *OC,* and in this plane describe a circle on *DE as* diameter. Through this circle describe a cylinder with axis *OC.*

This cylinder shall be the cylinder required, or its surface shall pass through every point *P* of the ellipse.

For, if *P* be any point on the ellipse, draw *PN* perpendicular to *AA';* through *N* draw *NM* parallel to *CO* meeting *DE* in *M,* and through *M,* in the plane of the circle on *DE* as diameter, draw *MQ* perpendicular to *DE,* meeting the circle in Q.

Then, since *DE = BB',*

*PN2 : AN • NA' =D02 : AC • CA'.*

And *DM • ME : AN • NA' =DO' • AC2,*since *AD, NM, CO, A'E* are parallel.

Therefore *PN2 = DM • ME*

*=012,*

by the property of the circle.

Hence, since *PN, QM* are equal as **Nvell** as parallel, *PQ* is parallel to MN and therefore to *CO.* It follows that PQ a generator of the cylinder, whose surface accordingly passes through *P.*

(2) If *BB' > DE,* we take *E'* on *A'E* such that *DE' =BB'* and describe a circle on *DE'* as diameter in a plane perpendicular to that of the paper; and the rest of the construction and proof is exactly similar to those given for case (1).

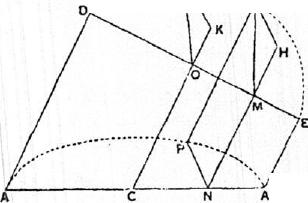
R (3) Suppose *BB' <DE.*

Take a point *K* on *CO* produced such

......

.••

that



*DO' — CB' =OK'.*

From *K* draw *KR* perpendicular to the plane of the paper and equal to *CB. Thus OR2=0K2+CB2=0D2.*

In the plane containing *DE, OR* de­scribe a circle on *DE* as diameter. Through this circle (which must pass through *R)* draw a cylinder with axis *OC.*

We have then to prove that, if *P* be

any point on t lie given ellipse, *P lice* on the cylinder so described.

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Draw *PN* perpendicular to *AA',* and through N draw *NM* parallel to *CO* meeting *DE* in M. In the plane of the circle on *DE* as diameter draw *MQ* perpendicular to *DE* and meeting the circle in Q.

Lastly, draw *QII* perpendicular to *NM* produced. *QH* will then be perpen­dicular to the plane containing *AC, DE,* i.e. the plane of the paper.

Now *QH2 : Q' = KR2 :OR', by* similar triangles.

And QM' : *AN NA' =DM • ME : AN • NA'*

*=0D2 : CA2.*

Hence, *ex aequali,* since *OR = OD,*

*QII2 : AN • NA' = KR2 :CA'*

*=CB2 :CA2*

*=PN2 : AN • NA'.*

Thus *QII =PN.* And *QII, PN* are also parallel. Accordingly *PQ* is parallel to

*MN,* and therefore to *CO,* so that *PQ* is a generator, and the cylinder passes

through *P.*

PROPOSITION 10

It was proved by the earlier geometers that any *two cones have to one another the ratio compounded of the ratios of their bases and of their heights.'* The same method of proof will show that *any segments of cones have to one another the ratio compounded of the ratios of their bases and of their heights.*

The proposition that *any `frustum' of a cylinder is triple of the conical segment which has the same base as the frustum and equal height* is also proved in the same manner as the proposition that *the cylinder is triple of the cone which has the same base as the cylinder and equal height.'*

PROPOSITION 11

(1) *If a paraboloid of revolution be cut by a plane through, or parallel to, the axis, the section will be a parabola equal to the original parabola which by its revolution generates the paraboloid. And the axis of the section will be the intersection between the cutting plane and the plane through the axis of the paraboloid at right angles to the cutting plane.*

*If the paraboloid be cut by a plane at right angles to its axis, the section will be a circle whose centre is on the axis.*

1. *If a hyperboloid of revolution be cut by a plane through the axis, parallel to the axis, or through the centre, the section will be a hyperbola, (a) if the section be through the axis, equal, (b) if parallel to the axis, similar, (c) if through the centre, not similar, to the original hyperbola which by its revolution generates the hyper-boloid. And the axis of the section will be the intersection of the cutting plane and the plane through the axis of the hyperboloid at right angles to the cutting plane.*

*Any section of the hyperboloid by a plane at right angles to the axis will be a circle whose centre is on the axis.*

1. *If any of the spheroidal figures be cut by a plane through the axis or parallel to the axis, the section will be an ellipse, (a) if the section be through the axis, equal,* (b) *if parallel to the axis, similar, to the ellipse which by its revolution gen-*

1This follows from Eucl. xII. 11 and 14 taken together. Cf. *On the Sphere and Cylinder z,* Lemma 1.

'This proposition was proved by Eudoxus, as stated in the preface to *On the Sphere* and *Cylinder* I. Cf. Eucl. xii. 10.

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*erates the figure. And the axis of the section will be the intersection of the cutting plane and the plane through the axis of the spheroid at right angles to the culling plane.*

*If the section be by a plane at right angles to the axis of the spheroid, it will be a circle whose centre is on the axis,*

(4) *If any of the said figures be cut by a plane through the axis, and if a per­pendicular be drawn to the plane of section from any point on the surface of the figure but not on the section, that perpendicular will fall within the section.*

"And the proofs of all these propositions are evident."

**PROPOSITION** 12

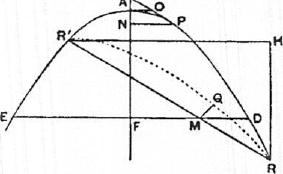
*If a paraboloid of revolution be cut by a plane neither parallel nor pendicular to the axis, and if the plane through the axis perpendicular to the cutting plane inter­sect it in a straight line of which the portion intercepted within the paraboloid is RR', the section of the paraboloid will be an ellipse whose major axis is RR' and whose minor axis is equal to the perpendicular distance between the lines through I?, I?' parallel to the axis of the paraboloid.*

Suppose the cutting plane to be perpendicular to the plane of the paper, and let the latter be the plane through the axis *ANF* of the paraboloid which inter­sects the cutting plane at right angles in *RR'.* Let *RH* be parallel to the axis of the paraboloid, and *R'H* perpendicular to *RH.*

Let *Q* be any point on the section made by the cutting plane, and from *Q* draw QM perpendicular to *RR'. QM* will therefore be perpendicular to the plane of the paper.

Through *M* draw *DMFE* perpendicular to the axis *ANF* meeting the para­bolic section made by the plane of the paper in *D, E.* Then *QM* is perpendicular to *DE,* and, if a plane be drawn through

**7** *DE, QM,* it will be perpendicular to the  
axis and will cut the paraboloid in a circu­lar section.



Since Q is on this circle,

*QM2 = DM • ME.*

Again, if *PT* be that tangent to the para­bolic section in the plane of the paper which is parallel to *RR',* and if the tangent at *A* meet *PT* in *0,* then, from the property of the parabola,

*DM • ME : RM . MR' =A02 :OP* [Prop. 3 (1)]  
*=A02 : OT2,* since AN = *A*7'.

Therefore *QM2* : *RM • MR' = A 02 : 0T2*

*= R'112 : RR",*

by similar triangles.

Hence *Q* lies on an ellipse whose major axis is *RI?'* and whose minor axis is

equal to *R'H.*

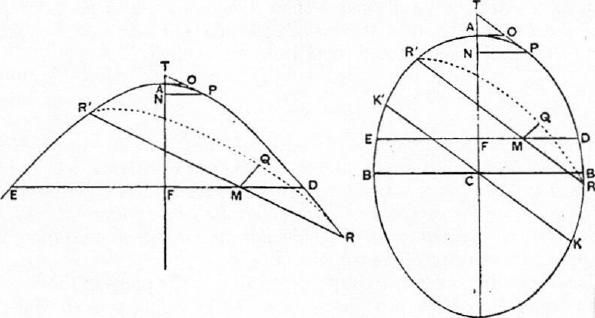
**PROPOSITIONS** 13, 14

*if* a *hyperboloid of revolution be cut by a plane meeting all the generators of the enveloping cone, or if an `oblong' spheroid be cut by a plane not perpendicular to*

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*the axis,' and if a plane through the axis intersect the cutting plane at right angles in a straight line on. which the hyperboloid or spheroid intercepts a length RR', then the section by the cutting plane will be an ellipse whose major axis is RR'.*

Suppose the cutting plane to be at right angles to the plane of the paper, and suppose the latter plane to be that through the axis *ANF* which intersects the



cutting plane at right angles in *RR'.* The section of the hyperboloid or spheroid by the plane of the paper is thus a hyperbola or ellipse having :INF for its transverse or major axis.

Take any point on the section made by the cutting plane, as Q, and draw QM perpendicular to *RR'. QM* will then be perpendicular to the plane of the paper.

Through M draw *DFE* at right angles to the axis ANY meeting the hyper­bola or ellipse in *D, E;* and through QM, *DE* let a plane be described. This plane **'Will** accordingly be perpendicular to the axis and will cut the hyperboloid or spheroid in a circular section.

Thus *QM2* =Dm • *ME.*

Let *PT* be that tangent to the hyperbola or ellipse which is parallel t o *RR',*

and let the tangent at *A* meet *PT* in 0.

Then, by the property of the hyperbola w•

*DM • ME • MR' =0.42 :OP',*

**or** Q.1/ : *RM. • MR' =OA' : OP'.*

Now (11 in the hyperbola *OA <OP,* because :17' <A N, and accordingly

*OT <OP,* while OA *<OT,*

(2) in the ellipse, if KK.' be the diameter parallel to *Bic,* and *DB'* the

minor axis,

*BC • CB' : KC • CK' =OA' :OP";*

and *BC • CB' <KC •CK' ,* so that OA <VI'.

Hence in both cases the locus of Q is an ellipse whose major axis is UR'.

**COR. 1.** If the spheroid be a `flat' spheroid, the section will be an ellipse, and

everything will proceed as before except that *RR'* will in this case he the

*minor* axis.

'Archimedes begins Prop. 14 for the *spheroid* with the remark that, when the cutting plane passes through or is parallel to the axis, the case is clear. Cf. Prop. 11 (3).

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CoR 2. In all conoids or spheroids parallel sections will be similar, since the ratio *0A2 : OP'* is the same for all the parallel sections.

PROPOSITION 15

(1) *If from any point on the surface of a conoid a line be drawn, in the case of the paraboloid, parallel to the axis, and, in the case of the hyperboloid, parallel to any line passing through the vertex of the enveloping cone, the part of the straight line which is in the same direction as the convexity of the surface will fall without it, and the part which is in the other direction within it.*

For, if a plane be drawn, in the case of the paraboloid, through the axis and the point, and, in the case of the hyperboloid, through the given point and through the given straight line drawn through the vertex of the enveloping cone, the section by the plane will be (a) in the paraboloid a parabola whose axis is the axis of the paraboloid, *(b)* in the hyperboloid a hyperbola in which the given line through the vertex of the enveloping cone is a diameter.' [Prop. 11] Hence the property follows from the plane properties of the conics.

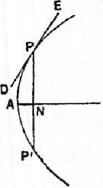
(2) *If a plane touch a conoid without cutting it, it will touch it at one point only, and the plane drawn through the point of contact and the axis of the conoid will be at right angles to the plane which touches it.*

For, if possible, let the plane touch at two points. Draw through each point a parallel to the axis. The plane passing through both parallels will therefore either pass through, or be parallel to, the axis. Hence the section of the conoid made by this plane will be a conic [Prop. 11 (1), (2)], the two points will lie on this conic, and the line joining them will lie within the conic and therefore within the conoid. But this line will be in the tangent plane, since the two points are in it. Therefore some portion of the tangent plane will be within the conoid; which is impossible, since the plane does not cut it.

Therefore the tangent plane touches in one point only.

That the plane through the point of contact and the axis is perpendicular to the tangent plane is evident in the particular case where the point of contact is the vertex of the conoid. For, if two planes through the axis cut it in two conics, the tangents at the vertex in both conics will be perpendicular to the axis of the conoid. And all such tangents will be in the tangent plane, which must therefore be perpendicular to the axis and to any plane through the axis.

If the point of contact *P* is not the vertex, draw the plane



passing through the axis *AN* and the point *P.* It will cut the conoid in a conic whose axis is *AN* and the tangent plane in a line *DPE* touching the conic at *P.* Draw *PNP'* perpendicular to the axis, and draw a plane through it also perpendicular to the axis. This plane will make a circular section and meet the tangent plane in a tangent to the circle, which will therefore be at right angles to *PN.* Hence the tangent to the circle will be at right angles to the plane containing *PN, AN;* and it follows that this last plane is perpendicular to the tangent plane.

1There seems to be some error in the text *here,* which says that "the *diameter"* (i.e. axis) of the hyperbola is "the straight line drawn in the conoid from the vertex of the cone." But this straight line is not, in general, the *axis* of the section.

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**PROPOSITION** 16

(1) *If a plane touch any of the spheroidal figures without cutting it, it will touch at one point only, and the plane through the point of contact and the axis will be at right angles to the tangent plane.*

This is proved by the same method as the last proposition.

1. *If any conoid or spheroid be cut by a plane through the axis, and if through any tangent to the resulting conic a plane be erected at right angles to the plane of section, the plane so erected will touch the conoid or spheroid in the same point as that in which the line touches the conic.*

For it cannot meet the surface at any other point. If it did, the perpendicular from the second point on the cutting plane would be perpendicular also to the tangent to the conic and would therefore fall outside the surface. But it must

fall within it. [Prop. 11 (4)]

1. *If two parallel planes touch any of the spheroidal figures, the line joining the points of contact will pass through the centre of the spheroid.*

If the planes are at right angles to the axis, the proposition is obvious. If not, the plane through the axis and one point of contact is at right angles to the tangent plane at that point. It is therefore at right angles to the parallel tan­gent plane, and therefore passes through the second point of contact. Hence both points of contact lie on one plane through the axis, and the proposition is reduced to a plane one.

**PROPOSITION** *17*

*If two parallel planes touch any of the spheroidal figures, and another plane be drawn parallel to the tangent planes and passing through the centre, the line drawn through any point of the circumference of the resulting section parallel to the chord of contact of the tangent planes will fall outside the spheroid.*

This is proved at once by reduction to a plane proposition.

Archimedes adds that it is evident that, if the plane parallel to the tangent planes does not pass through the centre, a straight line drawn in the manner described **\Sill** fall without the spheroid in the direction of the smaller segment but within it in the other direction.

**PROPOSITION** 18

Any *spheroidal figure which is cut by a plane through the centre is divided, both as regards its surface and its volume, into two equal parts by that plane.*

To prove this, Archimedes takes another equal and similar spheroid, divides it similarly by a plane through the centre, and then uses the method of application.

**PROPOSITIONS** 19, 20

*Given* a *segment cut off by a plane from a paraboloid or hyperboloid of revolution, or a segment of a spheroid less than half the spheroid also cut off by a plane, it is possible to inscribe in the segment one solid figure and to circumscribe about it another solid figure, each made up of cylinders or "frusta" of cylinders of equal height, and such that the circumscribed figure exceeds the inscribed figure by a volume less than that of any given solid.*

Let the plane base of the segment be perpendicular to the plane of the paper,

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and let the plane of the paper be the plane through the axis of the conoid or spheroid which cuts the base of the segment at right angles in *BC.* The section

in the plane of the paper is then a conic *BAC.* [Prop. 11]  
Let *EAF* be that tangent to the conic which is parallel to *BC,* and let *A* be the point of contact. Through *EAF* draw a plane parallel to the plane through *BC* bounding the segment. The plane so drawn will then touch the conoid or

spheroid at *A.* [Prop. 16]

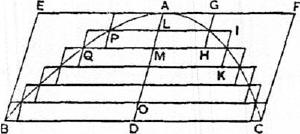
1. If the base of the segment is at right angles to the axis of the conoid or spheroid, A will be the vertex of the conoid or spheroid, and its axis AD will bisect *BC* at right angles.
2. If the base of the segment is not at right angles to the axis of the conoid or spheroid, we draw *AD*
3. in the paraboloid, parallel to the axis,
4. in the hyperboloid, through the cent re (or the vertex of the enveloping cone),
5. in the spheroid, through the centre,

and in all the cases it will follow that *AD* bisects *BC* in D.

Then *A* will be the vertex of the segment, and AD will be its axis.

Further, the base of the segment will be a circle or an ellipse with *BC* as diameter or as an axis respectively, and with centre *D.* We can therefore describe through this circle or ellipse a cylinder or a `frustum' of a cylinder

whose axis is *AD.* [Prop. 9]  
Dividing this cylinder or frustum continually into equal parts by planes parallel to the base, we shall at length arrive at a cylinder or frustum less in volume than any given solid.



Let this cylinder or frustum be that whose axis is OD, and let *AD* be divided into parts equal to *OD,* at *L, M* Through *L, M, .* draw lines parallel to *BC* meeting the conic in *P,* Q,. , and through these lines draw planes parallel to the base of the segment. These will cut the conoid or spheroid in circles or similar ellipses. On each of these circles or ellipses describe two cylinders or frusta of cylinders each with axis equal to *OD,* one of them lying in the direc­tion of *A* and the other in the direction of *D,* as shown in the figure.

Then the cylinders or frusta of cylinders drawn in the direction of A make up a circumscribed figure, and those in the direction of *D* an inscribed figure, in relation to the segment.

Also the cylinder or frustum *PG* in the circumscribed figure is equal to the cylinder or frustum *PH* in the inscribed figure, *QI* in the circumscribed figure is equal to *QK* in the inscribed figure, and so on.

Therefore, by addition,

(circumscribed fig.) = (inscr. fig.)-1- (cylinder or frustum whose axis is *OD).* But the cylinder or frustum whose axis is *OD* is less than the given solid figure; whence the proposition follows.

"Having set out these preliminary propositions, let us proceed to demon­strate the theorems propounded with reference to the figures."

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PROPOSITIONS 21, 22

*Any segment of a paraboloid of revolution is half as large again as the* cone or *segment of a cone which has the same base and the same axis.*

Let the base of the segment be perpendicular to the plane of the paper, and let the plane of the paper be the plane through the axis of the paraboloid which cuts the base of the segment at right angles in *BC* and makes the parabolic section *BAC.*

Let *EF* be that tangent to the parabola which is parallel to *BC,* and let A he the point of contact.

Then (1), if the plane of the base of the segment is perpendicular to the axis of the paraboloid, that axis is the line *AD* bisecting *BC* at right angles in *D.*

(2) If the plane of the base is not perpendicular to the axis of the paraboloid, draw *AD* parallel to the axis of the paraboloid. *AD will* then bisect *BC,* but not at right angles.

Draw through *EF* a plane parallel to the base of the segment. This will touch the paraboloid at *A,* and *A* will be the vertex of the segment, AD its axis.

The base of the segment will be a circle with diameter *BC* or an ellipse with *BC* as major axis.

Accordingly a cylinder or a frustum of a cylinder can be found passing through the circle or ellipse and having *AD* for its axis [Prop. 9]; and likewise a cone or a segment of a cone can be drawn passing through the circle or

ellipse and having A for vertex and *A* D for axis. [Prop. 8]  
Suppose X to be a cone equal to :1 (cone or segment of cone *ABC).* The cone X is therefore equal to half the cylinder or frustum of a cylinder *EC.* [Cf. Prop. 10; We shall prove that the volume of the segment of the paraboloid is equal to *X.*

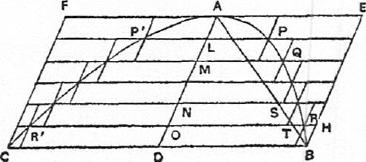
If not, the segment must be either greater or less than X.

I. If possible, let the segment be greater than X.

We can then inscribe and circumscribe, as in the last proposition, figures made up of cylinders or frusta of cylinders with equal height and such that. (circumscribed fig.) — (inscribed fig.) < (segment) — *X.*

Let the greatest of the cylinders or frusta forming the circumscribed figure be that whose base is the circle or ellipse about *BC* and whose axis is *OD,* and let the smallest of them be that whose base is the circle or ellipse about *PP'* and whose axis is *AL.*

Let the greatest of the cylin­ders forming the inscribed figure be that whose base is the circle or ellipse about *RR'* and whose axis is *OD,* and let the smallest be that whose base is the circle or ellipse about *PP'* and whose axis is *LM.*



Produce all the plane bases of the cylinders or frusta to meet the surface of the complete cylinder or frustum *EC.*

Now, since

(circumscribed fig.) — (inscr. fig.) < (segment)— *X ,*

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it follows that (inscribed figure)> X. (a)

Next, comparing successively the cylinders or frusta with heights equal to

*OD* and respectively forming parts of the complete cylinder or frustum *EC*

and of the inscribed figure, we have

(first cylinder or frustum in *EC) :* (first in inscr. fig.)

*w.B.D2* : *RO2*

*=AD :AO*

*= BD : TO,* where *AB* meets *OR* in T.

And (second cylinder or frustum in *EC) :* (second in inscr. *fig.)*

*=HO : SN,* in like manner,

and so on.

Hence [Prop. 1] (cylinder or frustum *EC) :* (inscribed figure)

*=(BD-1-H0+ • • •) : (TO-f-SN-1- • • -),*

where *BD, HO, ...* are all equal, and *BD, TO, SN, ...* diminish in arithmetical

progression.

But [Lemma preceding Prop. 1]

*BD-1-H0-1- • • •>2(T0-1--SN+ • • •).*

Therefore (cylinder or frustum *EC)>2* (inscribed fig.),

or *X>* (inscribed fig.);

which is impossible, by (a) above.

II. If possible, let the segment be less than *X.*

In this case we inscribe and circumscribe figures as before, but such that

(circumscr. fig.) —• (inscr. fig.) <*X--* (segment),

whence it follows that

(circumscribed figure) <*X.* (0)

And, comparing the cylinders or frusta making up the complete cylinder or

frustum *CE* and the *circumscribed* figure respectively, we have

(first cylinder or frustum in *CE) :* (first in circurnscr. fig.)

=BD2 : BD2

*=BD : BD.*

(second in *CE) :* (second in circumscr. fig.)

*=HO' : RO'*

*=AD :AO*

*=HO : TO,*

and so on.

Hence [Prop. I]

(cylinder or frustum *CE) :* (circumscribed fig.)

*\_(BD-I-H0-1- • • •) : (BD-1-T0+ .),*

*<2:1,* [Lemma preceding Prop. 1]

and it follows that

*X* < (circumscribed fig.);

which is impossible, by *(13).*

Thus the segment, being neither greater nor less than *X,* is equal to it, and

therefore to (cone or segment of cone *ABC).*

**PROPOSITION** 23

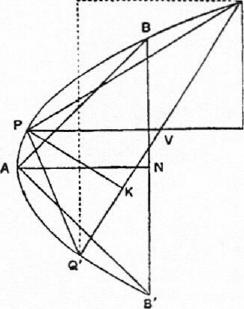
*If frotn* a *paraboloid of revolution two segments be cut off, one by a plane perpen­dicular to the axis, the other by a plane not perpendicular to the axis, and if the axes of the segments are equal, the segments will be equal in volume.*

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**Let the two planes be supposed perpendicular to the plane of the paper, and let the latter plane be the plane through the axis of the paraboloid cutting the other two planes at right angles in *BB', QQ'* respectively and the paraboloid itself in the parabola *QPQ'B'.***

**Let *AN, PV* be the equal axes of the seg­ments, and *A, P* their respective vertices.**

I.



**Draw *QL* parallel to *AN* or *PV* and *Q'L* perpendicular to *QL.***

**Now, since the segments of the parabolic section cut off by *BB',* QQ' have equal axes, the triangles *ABB', PQQ'* arc equal [Prop. 3]. Also, if *QD* be perpendicular to *PV,QD=BN* (as in the same Prop. 3).**

**Conceive two cones drawn with the same bases as the segments and with *A, P* as ver­tices respectively. The height of the cone *PQQ'* is then *PK,* where *PK* is perpendicular to *QQ'.***

**Now the cones are in the ratio compounded of the ratios of their bases and of their heights, i.e. the ratio compounded of (1) the ratio of the circle about *BB'* to the ellipse about QQ', and (2) the ratio of *AN* to *PK.***

**That is to say, we have, by means of Props. 5, 12,**

**(cone *ABB') :* (cone *PQQ') = (BB'2 : QQ' • Q'L) •(AN : PK).***

**And *BB' = 2BN = 2QD = Q'L,* while QQ' = 2QV.**

**Therefore**

**(cone *ABB') :* (cone *PQQ') = (QD :QV) •(AN : PK)***

***= (PK : PV) •(AN : PK)***

***= AN : PV.***

**Since *AN = PV,* the ratio of the cones is a ratio of equality; and it follows**

**that the segments, being each half as large again as the respective cones [Prop.**

**22], are equal.**

|  |  |
| --- | --- |
|  | **PROPOSITION 24**  ***If from a paraboloid of revolution two segments be cut off by planes drawn in* any *manner, the segments will be to one another as the squares on their axes.***  **For let the paraboloid be cut by a plane through the axis in the parabolic section *P'PApp',* and let the axis of the parabola and paraboloid he *ANN'.***  **Measure along *ANN'* the lengths *A N, AN'* equal to the respective axes of the given segments, and through *N, N'* draw planes perpendicular to the axis, making circular sections on *Pp, P'p'* as diameters respectively. With these circles as bases and with the common ver­tex *A* let two cones be described.**  **Now the segments of the paraboloid whose bases are the circles about *Pp, P'p'* are equal to the given segments respectively, since their respective axes are** |

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**equal [Prop. 23]; and, since the segments *APp, AP'p'* are half** as **large again**

**as the cones *APp, AP'p'* respectively, we have only to show that the cones**

**are in the ratio of *AN2* to *AN'2.***

**But**

**(cone *APp) :* (cone *AP'p') = (PN2 : P'N") -(AN : AN')***

***(AN : AN') •(AN : AN')***

***= AN2 : AN'2;***

**thus the proposition is proved.**

**PROPOSITIONS 25, 26**

***In any hyperboloid of revolution, if A be the vertex and AD the axis of any segment cut off by a plane, and if CA be the semidiameter of the hyperboloid through A (CA being of course in the same straight line with AD), then***

***(segment) : (cone with same base and axis)***

***= (AD-E3CA) : (AD-F2CA).***

**Let the plane cutting off the segment be perpendicular to the plane of the paper, and let the latter plane be the plane through the axis of the hyperboloid which intersects the cutting plane at right angles in *BB',* and makes the hyper­bolic segment *BAB'.* Let *C* be the centre of the hyperboloid (or the vertex of the enveloping cone).**

**Let *EF* be that tangent to the hyperbolic section which is parallel to *BB'.* Let *EF* touch at *A,* and join *CA.* Then *CA* produced will bisect *BB'* at *D, CA* will be a semi-diameter of the hyperboloid, *A* will be the vertex of the segment, and *AD* its axis. Produce *AC* to *A'* and *H,* so that *AC =CA' =A'H.***

**Through *EF* draw a plane parallel to the base of the segment. This plane will touch the hyperboloid at A.**

**Then (1), if the base of the segment is at right angles to the axis of the hyperboloid, *A* will be the vertex, and *AD* the axis, of the hyperboloid as well as of the segment, and the base of the segment will be a circle on *BB'* as diameter.**

**(2) If the base of the segment is not perpendicular to the axis of the hyper-**

**boloid, the base will be an ellipse on *BB'* as major axis. [Prop. 13]**

**Then we can draw a cylinder or a frustum of a cylinder *EBB' F* passing through the circle or ellipse about *BB'* and having *AD* for its axis; also we can describe a cone or a segment of a cone through the circle or ellipse and having A for its vertex.**

**We have to prove that**

**(segment *ABB') :* (cone or segment of cone *ABB') = HD : A'D.* Let V be a cone such that**

***V :* (cone or segment of cone *ABB') =HD : A'D, (a)***

**and we have to prove that *V* is equal to the segment.**

***Now***

**(cylinder or frustum *EB') :* (cone or segmt. of cone *ABB') =3* : 1.**

**Therefore, by means of (a), (cylinder or frustum *EB') : V= A' D : HD*** *((3)*

**If the segment is not equal to *V,* it must either be greater or less.**

**I. If possible, let the segment be greater than V.**

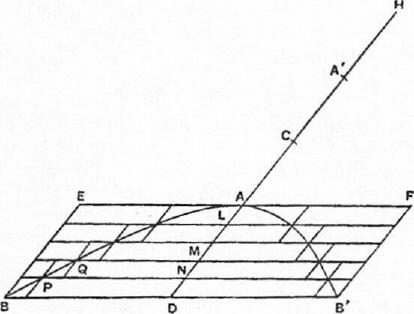
**Inscribe and circumscribe to the segment figures made up of cylinders or**

**frusta of cylinders, with axes along *AD* and all equal to one another, such that**

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**(circumscribed fig.) — (inscr. fig.) < (segmt.) — *V,***

**whence (inscribed figure) > *V.* (7)  
Produce all the planes forming the bases of the cylinders or frusta of cylin-**  
**ders to meet the surface of the complete cylinder or frustum *EB'.***



p

(AD) p q

(A A') **a a a**

a

b

$

**Then, if *ND* be the axis of the greatest cylinder or frustum in the circum­scribed figure, the complete cylinder will be divided into cylinders or frusta each equal to this greatest cylinder or frustum.**

**Let there be a number of straight lines a equal to *AA'* and as many in num­ber as the parts into which *AD* is divided by the bases of the cylinders or frus­ta. To each line a apply a rectangle which shall overlap it by** a **square, and let**

**the greatest of the rectangles be equal to the rectangle *AD* and the least  
equal to the rectangle *AL •A'L;* also let the sides of the overlapping squares *b, p, q,.. .1* be in descending arithmetical progression. Thus h, p, *q,. . .1* will be respectively equal to *AD, AN, AM,. . . AL,* and the rectangles *(ab-+b2),* (ap-Fp2),. *(a1+12)* will be respectively equal to *AD -A'D, AN •A'N,. . AL A'L.***

**Suppose, further, that we have a series of spaces *S* each equal to the largest rectangle *AD •A'D* and as many in number as the diminishing rectangles. Comparing now the successive cylinders or frusta (1) in the complete cylin-**

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der or frustum *EB'* and (2) in the inscribed figure, beginning from the base of

the segment, we have

(first cylinder or frustum in *£B') :* (first in inscr. figure)

*=BD2* : *PN2*

*=AD •A'D : AN •A'N,* from the hyperbola,

*=8 : (ap-Fp2).*

Again

(second cylinder or frustum in *EB') :* (second in inscr. fig.)

*=BD2* : *QM2*

*=AD •A'D : AM •A'M*

*=8 :* (aq+q2),

and so on.

The last cylinder or frustum in the complete cylinder or frustum *EB'* has no

cylinder or frustum corresponding to it in the inscribed figure.

Combining the proportions, we have [Prop. 1]

(cylinder or frustum *EB') :* (inscribed figure)

= (sum of all the spaces *S) : (api-p2)+(aq+0)-1- • • •*

*> (a+b) :* (22 -1- 120 [Prop. 2]

*D*

*H*

***>A' D :*** —3—, since *a=AA' , b= AD,*

*> (EB') : V,* by (0) above.

Hence (inscribed figure) < *V.*

But this is impossible, because, by (7) above, the inscribed figure is greater

than V

II. Next suppose, if possible, that the segment is less than *V.*

In this case we circumscribe and inscribe figures such that

(circumscribed fig.) — (inscribed fig.) < *V—* (segment),

whence we derive

*V>* (circumscribed figure). (5)

We now compare successive cylinders or frusta in the complete cylinder or

frustum and in the *circumscribed* figure; and we have

(first cylinder or frustum in *EB') :* (first in circumscribed fig.)

*=S : S*

*=S : (ab+b2),*

(second in *EB') :* (second in circumscribed fig.)

=8 *:* (ap+p2),

|  |  |
| --- | --- |
| and so on.  Hence [Prop. 1]  (cylinder or frustum *EB') :* (circumscribed fig.)  = (sum of all spaces ***S) :*** (ab+b2)+(ap-1-732)+ *• • •*  *<(a+b) : (*2*c14)*3  *<ii'D :D*  *—3—*  *< (EB') : V, by (i3)* above. | [Prop. 2] |

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Hence the circumscribed figure is greater than *V;* which is impossible, by

*(6)* above.

Thus the segment is neither greater nor less than *V,* and is therefore equal

to it.

Therefore, by (a),

(segment *ABB') :* (cone or segment of cone *ABB')*

*=(AD+3CA) : (AD+2CA).*

**PROPOSITIONS** 27, 28, 29, 30

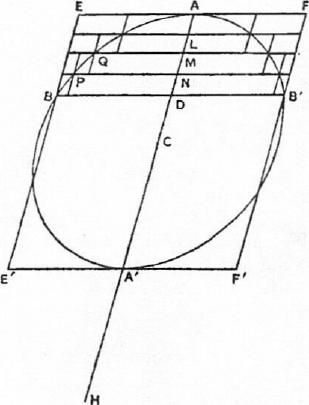
*(1) In any spheroid whose centre is C, if* a *plane meeting the axis cut off a segment not greater than half the spheroid and having A for its vertex and AD for its axis, and if A'D be the axis of the remaining segment of the spheroid, then*

*(first segmt.) : (cone or segmt. of cone with same base* and *axis) =CAI-A'D : A'D*

*[=3CA—AD :2CA—AD].*

(2) *As a particular case, if the plane passes through the centre, so that the seg­ment is half the spheroid, half the spheroid is double of the cone or segment of a cone which has the same vertex and axis.*

Let the plane cutting off the segment be at right angles to the plane of the paper, and let the latter plane be the plane through the axis of the spheroid which intersects the cut­ting plane in *BB'* and makes the elliptic section *ABA'B'.*



|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| **a'** | | | | | **a'** | | | **a'** | | |
| **‘,..**  **I** |  | **a** |  | | **a** |  | |  |
|  | **,\_..., x** |  |  | |  |  | | **a S** |
|  | |
|  | |
| ***d d d*** | | | | | | | | |

Let *EF, E'F'* be the two tan­gents to the ellipse which are par­allel to *BB',* let them touch it in *A, A',* and through the tangents draw planes parallel to the base of the segment. These planes will touch the spheroid at *A, A',* which will be the vertices of the two seg­ments into which it is divided. Also *AA'* will pass through the centre *C* and bisect *BB'* in *D.*

Then (1) if the base of the seg­ments be perpendicular to the axis of the spheroid, *A, A'* will be the vertices of the spheroid as well as of the segments, *AA'* will be the axis of the spheroid, and the base of the segments will be a circle on *BB'* as diameter;

(2) if the base of the segments be not perpendicular to the axis of the spheroid, the base of the segments will be an ellipse of which *BB'* is one axis, and *AD, A'D* will be the axes of the segments respec­tively.

ON CONOIDS AND SPHEROIDS 75

We can now draw a cylinder or a frustum of a cylinder ***EBB'?*** through the circle or ellipse about ***BB'*** and having ***AD*** for its axis; and we can also draw a cone or a segment of a cone passing through the circle or ellipse about ***BB'*** and having A for its vertex.

We have then to show that, if ***CA'*** be produced to H so that ***CA' =All,***

(segment ***ABB') :*** (cone or segment of cone ***ABB') = HD : A'D.*** Let ***V*** be such a cone that

***V :*** (cone or segment of cone ***ABB')= HD : A'D; (a)***

and we have to show that the segment ***ABB'*** is equal to ***V.***

But, since

(cylinder or frustum ***EB') :*** (cone or segment of cone ***ABB') = 3*** : I, we have, by the aid of (a),

***D H***

(cylinder or frustum ***EB') :V =A'D : —3*** (0)

Now, if the segment ***ABB'*** is not equal to V, it must be either greater or less.

I. Suppose, if possible, that the segment is greater than ***V.***

Let figures be inscribed and circumscribed to the segment consisting of

cylinders or frusta of cylinders, with axes along ***AD*** and all equal to one

another, such that

(circumscribed fig.) — (inscribed fig.) < (segment) — *V,*

whence it follows that

(inscribed fig.) > ***V. (7)***

Produce all the planes forming the bases of the cylinders or frusta to meet the surface of the complete cylinder or frustum ***EB'.*** Thus, if ***ND*** be the axis of the greatest cylinder or frustum of a cylinder in the circumscribed figure, the complete cylinder or frustum ***EB'*** will be divided into cylinders or frusta of cylinders each equal to the greatest of those in the circumscribed figure.

Take straight lines ***da'*** each equal to A'D and as many in number as the parts into which ***AD*** is divided by the bases of the cylinders or frusta, and measure da along ***da'*** equal to ***AD.*** It follows that aa' *= 2CD.*

Apply to each of the lines ***a'd*** rectangles with height equal to ***ad,*** and draw the squares on each of the lines ***ad*** as in the figure. Let ***S*** denote the area of each complete rectangle.

From the first rectangle take away a gnomon with breadth equal to ***AN*** (i.e. with each end of a length equal to ***AN);*** take away from the second rectangle a gnomon with breadth equal to ***AM,*** and so on, the last rectangle having no gnomon taken from it.

Then

the first gnomon = A'D •AD ***—ND -(A'D —AN)***

***=A'D -AN-FAT •AN***

***=AN •A'N.***

Similarly,

the second gnomon *=AM* ***•A'M,***

and so on.

And the last gnomon (that in the last rectangle but one) is equal to ***AL -A/L.***

Also, after the gnomons are taken away from the successive rectangles, the remainders (which we will call RI, R2, • • • ***R.,*** where n is the number of rec­tangles and accordingly ***R„= 2)*** are rectangles applied to straight lines each of

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length aa' and "exceeding by squares" whose sides are respectively equal to

***DN, DM,. . . DA.***

For brevity, let ***DN*** be denoted by *x,* and aa' or ***2CD*** by c, so that

Ri *=CX+X2,* **R2 *= c .2x (2x)2, • • •***

Then, comparing successively the cylinders or frusta of cylinders (1) in the

complete cylinder or frustum ***EB'*** and (2) in the inscribed figure, we have

(first cylinder or frustum in ***EB') :*** (first in inscribed fig.)

***=BD' :PN2***

***=AD •A'D : AN •A'N***

***=8 :*** (first gnomon);

(second cylinder or frustum in ***EB') :*** (second in inscribed fig.)

***=S :*** (second gnomon),

and so on.

The last of the cylinders or frusta in the cylinder or frustum ***ER'*** has none

corresponding to it in the inscribed figure, and there is no corresponding

gnomon.

Combining the proportions, we have [by Prop. 1]

(cylinder or frustum ***ER') :*** (inscribed fig.)

= (sum of all spaces *S)* : (sum of gnomons).

Now the differences between ***S*** and the successive gnomons are RI, R2,

while

*R1=CX+22,*

**R2= C '2x+ *(2x)2,***

***1?„=cb-Fb2=8,***

where b = ***nx= AD.***

Hence [Prop. 2]

(sum of all spaces ***S) : -1-R2+ • • • -F-R7,) <(c+b) :*** (2+0• It follows that

*c* 2b

(sum of all spaces *8)* : (sum of gnomons)>(c+b) ***:*** )

***>A'D :--3D.***

Thus (cylinder or frustum ***EB') :*** (inscribed fig.) ***>A'D HD***

> (cylinder or frustum ***EB') : V,***

from *(0)* above.

Therefore (inscribed fig.) < ***V;***

which is impossible, by (7) above.

Hence the segment ***ABB'*** is not greater than ***V.***

II. If possible, let the segment ***ABB'*** be less than ***V.***

We then inscribe and circumscribe figures such that

(circumscribed fig.) — (inscribed fig.) < ***V —*** (segment),

whence ***V>*** (circumscribed fig.). (8)  
In this case we compare the cylinders or frusta in ***(ER')*** with those in the  
***circumscribed figure.***

ON CONOIDS AND SPHEROIDS *77*

Thus

(first cylinder or frustum in *EB') :* (first in circumscribed fig.)

=S *: S;*

(second in *ER') :* (second in circumscribed fig.)

*:* (first gnomon),

and so on.

Lastly (last in *EB') :* (last in circumscribed fig.)

*: (last* gnomon).

Now

18-1- (all the gnomons)} = *nS (RI+ RI+ • • • +1?„\_11*

And *nS : R1+R2+ • • • +R.-1> (c+b) (2A),* [Prop. 2]

so that

c 2b

nS : {S+ (all the gnomons)} < *(c+b) : (-2+-3).*

It follows that, if we combine the above proportions as in Prop. 1, we obtain (cylinder or frustum *ER') :* (circumscribed fig.)

< (c+b) : (--2)

2 3

*<ND: HD*

3

*<(EB') : V,* by (j3) above.

Hence the circumscribed figure is greater than *V;* which is impossible, by (3)

above.

Thus, since the segment *ABB'* is neither greater nor less than *V,* it is equal

to it; and the proposition is proved.

(2) The particular case [Props. 27, 28] where the segment is half the spheroid

differs from the above in that the distance *CD* or c/2 vanishes, and the rec-

tangles *cb+b2* are simply squares (b2), so that the gnomons are simply the

differences between b2 and x2, b2 and (2x)2, and so on.

Instead therefore of Prop. 2 we use the *Lemma to Prop.* 2, *Cor.* 1, given above

*[On Spirals,* Prop. 10], and instead of the ratio *(c+b) : (-c*2*+ —*2b we obtain the

3

ratio 3 : 2, whence (segment *ABB') :* (cone or segment of cone *ABB')=2* : I. PROPOSITIONS 31, 32

*If* a *plane divide a spheroid into two unequal segments, and if AN, A'N be the axes of the lesser and greater segments respectively, while C is the centre of the spheroid, then*

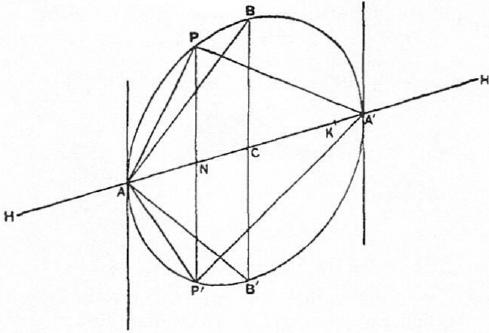
*(greater segmt.) : (cone or segmt. of cone with same base and axis)  
=CA+AN : AN.*

Let the plane dividing the spheroid be that through *PP'* perpendicular to the plane of the paper, and let the latter plane be that through the axis of the spheroid which intersects the cutting plane in *PP'* and makes the elliptic section *PAP'A'.*

Draw the tangents to the ellipse which are parallel to ***PP';*** let them touch the ellipse at *A,* ***A',*** and through the tangents draw planes parallel to the base of the segments. These planes will touch the spheroid at *A, A',* the line *AA'*

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will pass through the centre *C* and bisect *PP'* in *N,* while *AN, A'N* will be the axes of the segments.



Then (1) if the cutting plane be perpendicular to the axis of the spheroid,

A A' will be that axis, and *A, A'* will be the vertices of the spheroid as well as of

the segments. Also the sections of the spheroid by the cutting plane and all

planes parallel to it will be circles.

(2) If the cutting plane be not perpendicular to the axis, the base of the

segments will be an ellipse of which *PP'* is an axis, and the sections of the

spheroid by all planes parallel to the cutting plane will be similar ellipses.

Draw a plane through *C* parallel to the base of the segments and meeting the

plane of the paper in *BB'.*

Construct three cones or segments of cones, two having A for their common

vertex and the plane sections through *PP', BB'* for their respective bases, and

a third having the plane section through *PP'* for its base and A' for its vertex.

Produce *CA* to *H* and *CA'* to *H'* so that

*AH=A'H'*=CA.

We have then to prove that

(segment *A'PP') :* (cone or segment of cone *A'PP')*

*=CA-FAN : AN*

*=NH : AN.*

Now half the spheroid is double of the cone or segment of a cone *ABB'*

[Props. 27, 281. Therefore

(the spheroid) =4(cone or segment of cone *A BB').*

But

(cone or segmt. of cone *ABB') :* (cone or segmt. of cone *APP')*

*= (CA : AN) • (BC2 : PAT')*

*= (CA : AN) • (CA • CA' : AN • AR).* (a)

If we measure .4K along *AA'* so that

*AK : AC = AC :* AN,

we **have** *AK •* A'N : *AC • A'N =CA* : *AN,*

and the compound ratio in (a) becomes

*(AK • A'N : CA • A'N) • (CA • CA' : AN • A'N),*

i.e. *AK •* CA' : *AN • A'N.*

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Thus

(cone or segmt. of cone ***ABB') :*** (cone or segmt. of cone ***APP')***

***=AK • CA' : AN • A'N.***

But (cone or segment of cone ***APP') :*** (segment ***API")***

***=A'N : NH'*** [Props. 29, 301

***=AN • A'N : AN • NH'.***

Therefore, ***ex aequali,***

(cone or segment of cone ***ABB') :*** (segment ***API")***

***=AK • CA' : AN • NH',***

so that (spheroid) : (segment ***APP')***

***=HH' • AK : AN • NH',***

since ***HH' =4CA'.***

Hence (segment ***A'PP') :*** (segment ***APP')***

***=(HH' • AK—AN - NH') : AN • NH'***

***= (AK • NH-}-NH' • NK) : AN • NH'.***

Further,

(segment ***APP') :*** (cone or segment of cone ***APP')***

***=NH' : A'N***

***=AN • NH' AN • A'N,***

and

(cone or segmt. of cone ***APP') :*** (cone or segmt. of cone ***A'PP')***

***=AN : A'N***

***=AN • A'N : A'N'.***

From the last three proportions we obtain, ***ex aequali,***

(segment ***A'PP') :*** (cone or segment of cone ***A'PP')***

***= (AK • NH-I-NH' • NK) : A'N2***

***= (AK • NII+NH' • NK) : (CA2-1-NH' • CN)***

***=(AK • NHA-NH' • NK) : (AK - • CN). (13)***

But

***AK •NH : AK •AN =NH :AN***

***=CA+AN : AN***

***=AK+CA :CA*** (since ***AK : AC = AC : AN)***

***=HK :CA***

***=HK—NH :CA—AN***

***=NK : CN***

***=NH' • NK :NH' • CN.***

Hence the ratio in *(0)* is equal to the ratio

***AK•NH : AK • AN,*** or ***NH : AN.***

Therefore

(segment ***A'PP') :*** (cone or segment of cone ***A'PP')***

***=NH : AN***

***=CA+AN : AN.***

**ON SPIRALS**

**"ARCHIMEDES** to Dositheus greeting.

"Of most of the theorems which I sent to Conon, and of which you ask me from time to time to send you the proofs, the demonstrations are already be­fore you in the books brought to you by Heracleides; and some more are also contained in that which I now send you. Do not be surprised at my taking a considerable time before publishing these proofs. This has been owing to my desire to communicate them first to persons engaged in mathematical studies and anxious to investigate them. In fact, how many theorems in geometry which have seemed at first, impracticable are in time successfully worked out! Now Conon died before he had sufficient time to investigate the theorems referred to; otherwise he would have discovered and made manifest all these things, and would have enriched geometry by many other discoveries besides. For I know well that it was no common ability that he brought to bear on mathematics, and that his industry was extraordinary. But, though many years have elapsed since Conon's death, I do not find that any one of the prob­lems has been stirred by a single person. I wish now to put them in review one by one, particularly as it happens that there are two included among them which are impossible of realisation [and which may serve as a warning] how those who claim to discover everything but produce no proofs of the same may be confuted as having actually pretended to discover the impossible.

"What are the problems I mean, and what are those of which you have already received the proofs, and those of which the proofs are contained in this book respectively, I think it proper to specify. The first of the problems was, Given a sphere, to find a plane area equal to the surface of the sphere; and this was first made manifest on the publication of the book concerning the sphere, for, when it is once proved that the surface of any sphere is four times the greatest circle in the sphere, it is clear that it is possible to find a plane area equal to the surface of the sphere. The second was, Given a cone or a cylinder, to find a sphere equal to the cone or cylinder; the third, To cut a given sphere by a plane so that the segments of it have to one another an assigned ratio; the fourth, To cut a given sphere by a plane so that the segments of the surface have to one another an assigned ratio; the fifth, To make a given segment of a sphere similar to a given segment of a sphere;' the sixth, Given two segments of either the same or different spheres, to find a segment of a sphere which shall be similar to one of the segments and have its surface equal to the surface of the other segment. The seventh was, From a given sphere to cut off a seg­ment by a plane so that the segment bears to the cone which has the same base

*'Cf. On the Sphere and Cylinder, II.* 5.

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as the segment and equal height an assigned ratio greater than that of three to two. Of all the propositions just enumerated Heracleides brought you the proofs. The proposition stated next after these was wrong, viz. that, if a sphere be cut by a plane into unequal parts, the greater segment. will have to the less the duplicate ratio of that which the greater surface has to the less. That this is wrong is obvious by what I sent you before; for it included this proposition: If a sphere be cut into unequal parts by a plane at right angles to any diameter in the sphere, the greater segment of the surface will have to the less the same ratio as the greater segment of the diameter has to the less, while the greater segment of the sphere has to the less a ratio less than the duplicate ratio of that which the greater surface has to the less, but greater than the sesqui-alteratel of that ratio. The last of the problems was also wrong, viz. that, if the diameter of any sphere be cut so that the square on the greater segment is triple of the square on the lesser segment, and if through the point thus arrived at, a plane be drawn at right angles to the diameter and cutting the sphere, the figure in such a form as is the greater segment of the sphere is the greatest of all the segments which have an equal surface. That this is wrong is also clear from the theorems which I before sent you. For it was there proved that the hemisphere is the greatest of all the segments of a sphere bounded by an equal surface.

"After these theorems the following were propounded concerning the cone.' If a section of a right-angled cone [a parabola], in which the diameter [axis] remains fixed, be made to revolve so that the diameter [axis] is the axis [of revolution], let the figure described by the section of the right-angled cone be called a *conoid.* And if a plane touch the conoidal figure and another plane drawn parallel to the tangent plane cut off a segment of the conoid, let the *base* of the segment cut off be defined as the cutting plane, and the *vertex* as the point in which the other plane touches the conoid. Now, if the said figure be cut by a plane at right angles to the axis, it is clear that the section will be a circle; but it needs to be proved that the segment cut off will be half as large again as the cone which has the same base as the segment and equal height. And if two segments be cut off from the conoid by planes drawn in any man­ner, it is clear that the sections will he sections of acute-angled cones [ellipses] if the cutting planes be not at right angles to the axis; but it needs to be proved that the segments will bear to one another the ratio of the squares on the lines drawn from their vertices parallel to the axis to meet the cutting planes. The proofs of these propositions are not yet sent to you.

"After these came the following propositions about the *spiral,* which are as it were another sort of problem having nothing in common with the foregoing; and I have written out the proofs of them for you in this book. They are as follows. If a straight line of which one extremity remains fixed be made to revolve at a uniform rate in a plane until it returns to the position from which it started, and if, at the same time as the straight line revolves, a point move at a uniform rate along the straight line, starting from the fixed extremity, the point will describe a spiral in the plane. I say then that the area bounded by the spiral and the straight line which has returned to the position from which it started is a third part of the circle described with the fixed point as centre and with radius the length traversed by the point along the straight line during

'See *On the Sphere and Cylinder, II.* 8.

'This should be presumably "the *conoid,"* not "the cone."

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the one revolution. And, if a straight line touch the spiral at the extreme end of the spiral, and another straight line be drawn at right angles to the line which has revolved and resumed its position from the fixed extremity of it, so as to meet the tangent, I say that the straight line so drawn to meet it is equal to the circumference of the circle. Again, if the revolving line and the point moving along it make several revolutions and return to the position from which the straight line started, I say that the area added by the spiral in the third revolution will be double of that added in the second, that in the fourth three times, that in the fifth four times, and generally the areas added in the later revolutions will be multiples of that added in the second revolution according to the successive numbers, while the area bounded by the spiral in the first revolution is a sixth part of that added in the second revolution. Also, if on the spiral described in one revolution two points be taken and straight lines be drawn joining them to the fixed extremity of the revolving line, and if two circles be drawn with the fixed point as centre and radii the lines drawn to the fixed extremity of the straight line, and the shorter of the two lines be produced, I say that (1) the area bounded by the circumference of the greater circle in the direction of (the part of) the spiral included between the straight lines, the spiral (itself) and the produced straight line will bear to (2) the area bounded by the circumference of the lesser circle, the same (part of the) spiral and the straight line joining their extremities the ratio which (3) the radius of the lesser circle together with two thirds of the excess of the radius of the greater circle over the radius of the lesser bears to (4) the radius of the lesser circle together with one third of the said excess.

"The proofs then of these theorems and others relating to the spiral are given in the present book. Prefixed to them, after the manner usual in other geometrical works, are the propositions necessary to the proofs of them. And here too, as in the books previously published, I assume the following lemma, that, if there be (two) unequal lines or (two) unequal areas, the excess by which the greater exceeds the less can, by being [continually] added to itself, be made to exceed any given magnitude among those which are comparable with [it and with] one another."

**PROPOSITION 1**

*If* a *point move at a uniform rate along any line, and two lengths be taken on it, they will be proportional to the times of describing them.*

Two unequal lengths are taken on a straight line, and two lengths on another straight line representing the times; and they are proved to be proportional by taking equimultiples of each length and the corresponding time after the man­ner of Eucl. V, Def. 5.

**PROPOSITION** 2

*If each of two points on different lines respectively move along them each at a uni­form rate, and if lengths be taken, one* on each line, *forming* pairs, *such that* each *pair are described in equal times, the lengths will be porportionals.*

This is proved at once by equating the ratio of the lengths taken on one line to that of the times of description, which must also be equal to the ratio of the lengths taken on the other line.

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PROPOSITION 3

*Given any number of circles, it is possible to find a straight line greater than the sum of all their circumferences.*

For we have only to describe polygons about each and then take a straight line equal to the sum of the perimeters of the polygons.

PROPOSITION 4

*Given two unequal lines, viz. a straight line and the circumference of a circle, it is possible to find a straight line less than the greater of the two lines and greater than the less.*

For, by the Lemma, the excess can, by being added a sufficient number of times to itself, be made to exceed the lesser line.

Thus e.g., if *c> 1* (where *c* is the circumference of the circle and *1* the length of the straight line), we can find a number n such that

*n(c—l)>l.*

Therefore *c-1>*n-/'

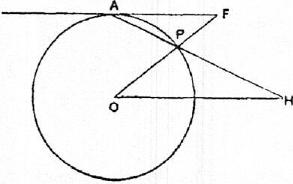
and c>/-1--n-/>/.

Hence we have only to divide *1* into n equal parts and add one of them to *1.* The resulting line \vitt satisfy the condition.

PROPOSITION 5

*Given* a *circle with centre* 0, *and the tangent to it at a point A, it is possible to draw from 0 a straight line OPF, meeting the circle in P and the tangent in. F, such that, if c be the circumference of any given circle whatever,*

*FP:OP<* (arc *AP) : c.*



Take a straight line, as *D,* greater than the circumference c. [Prop. 3]

Through *0* draw *OH* parallel to the given tangent, and draw through A a line *APH,* meeting the circle in *P* and *OH* in *H,* such that the portion *PH* intercepted between the circle and the line *OH* may he equal to D. Join *OP* and produce it to meet the tangent in *F.*

Then *FP :OP=AP : PH,* by parallels,

*D* = *AP : D*

< (arc *AP) : c.*

PROPOSITION 6

*Given a circle with centre 0, a chord AB less than the diameter, and OM the per-*

*pendicular on AB from 0, it is possible to draw a straight line OFP, meeting the*

*chord AB in F and the circle in P, such that*

*FP :PB=D :E,*

*where D : E is any given ratio less than BM :* MO.

Draw *011* parallel to *AB,* and *B7'* perpendicular to *BO* meeting *011* in 7'.

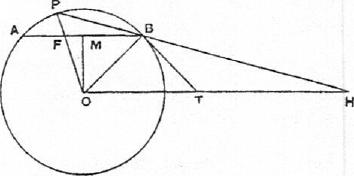
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Then the triangles *BRIO, OBT* are similar, and therefore

*BM : ilfO =OB : BT,*

whence *D : E <OR : 1371.*

Suppose that a line *PH* (greater than *BT)* is taken such that



D

*D :E=OB :PH,*

and let *PH* be so placed that it passes through *B* and *P* lies on the circumference of the circle, while *H* is on the line *OH. (PH* will fall outside *BT,* because *PH>BT.)* Join *OP* meeting *AB* in *F.*

We now have

*FP :PB=OP :PH*

*=OB :PH*

*=D : E.*

PROPOSITION *7*

*Given a circle with centre 0, a chord AB less than the diameter, and OM the per-*

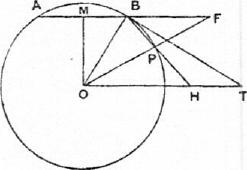
*pendicular on it from 0, it is possible to draw from 0 a straight line OPF, meeting*

*the circle in P and AB produced in F, such that*

*FP :PB=D :E,*

*where D : E is any given ratio greater than BM :MO.*

Draw *OT* parallel to *AB,* and *BT* perpendic-



0

E

ular to *BO* meeting *OT* in *T.*

In this case,

*D : E> BM : MO*

*> OB : BT,* by Fimilar triangles.

Take a line *PH* (less than *BT)* such that

*D : E=OB :PH,*

and place *PH* so that *P, II* are on the circle and

on *OT* respectively, while *HP* produced passes

through *B.*

Then *FP :PB=OP : PH*

*=D : E.*

PROPOSITION 8

*Given a circle with centre 0, a chord AB less than the diameter, the tangent at B,*

*and the perpendicular OM from 0 on AB, it is possible to draw from 0* a *straight*

*line OFP, meeting the chord AB in F, the circle in P and the tangent in G, such*

*that*

*FP : BG = D : E,*

*where D : E is any given ratio less than BM : MO.*

If *OT* be drawn parallel to *AB* meeting the tangent *at B* in *T,*

*BM :2110=0B : BT,*

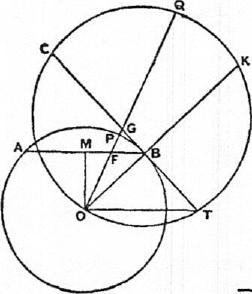
so that *D : E <OB : BT.*

Take a point *C* on *TB* produced such that

D : *E= OB : 13C,*

whence *BC> BT.*

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Through the points *0, T,C* describe a circle, and let *OB* be produced to meet this circle in *K.*

Then, since *BC>BT,* **and** *013* is perpendicular to *CT,* it is possible to draw from *0* a straight line *OGQ,* meeting *CT* in *G* and the circle about *OTC* in Q, such that *GQ=BK.*

Let *OGQ* meet *AB* in *F* and the original circle in *P.*

Now *CG • GT =OG • GQ;*

**o** and *OF :0G=BT :GT,*

E  **SO** that *OF • GT =OG • BT.*

It follows that

*CG • GT :OF • GT =OG • GQ : OG • BT,*

**or** *CG :OF =GQ :B7'*

*= BK : BT,* by construction,

*=BC : OB*

*=BC :OP.*

Hence *OP : OF =BC :CG,*

and therefore *PF : OP= BG : BC,*

**or** *PF : BG =OP : BC*

*=OB : BC  
=D :E.*

**PROPOSITION** 9

*Given a circle with centre 0, a chord AB less than. the diameter, the tangent at B,*

*and the perpendicular OM from 0 on AB, it is possible to draw from 0 a straight*

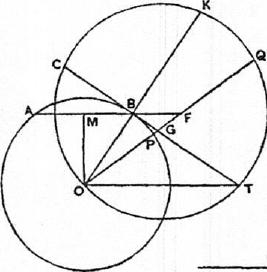
*line OPGF, meeting the circle in P, the tangent in G, and AB produced in F,*

*such that*

*FP : BG =D :E,*

*where D : E is any given ratio greater than BM : MO.*

Let *OT* be drawn parallel to *AB*



meeting the tangent at *B* in *T.*

Then

D *E> BM : MO*

*> OB : BT,* by similar triangles.

Produce *TB* to *C* so that

*D :E=OB : BC,*

whence *BC<BT.*

Describe a circle through the

**points** *0, T, C,* and produce *OB* to

meet this circle in *K.*

Then, since *TB>BC,* and *OB* is

perpendicular to *CT,* it is possible to

draw from *0* a line *OGQ,* meeting

*CT* in *G,* and the circle about *OTC in Q,* such that *GQ= BK.* Let *OQ* meet the *original* circle in *P* **and *AB*** pro­duced **in** *F.*

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We now prove, exactly as in the last proposition, that

*CG :OF = BK :BT*

*=BC :OP.*

Thus, as before,

*OP :OF =BC :CG,*

and *OP : PF = BC : BG,*

whence *PF : BG=OP : BC*

*=OB : BC  
=D : E.*

PROPOSITION 10

*If A1, A2, A3,* **• •** *•An be n lines forming an ascending arithmetical progression in which the common difference is equal to A1, the least term, then*

(n-1-1).An2-FAI(Aid-A2+ • • • *+A„)=3(4412-1-A22-1- • • •-1-21„2).* [Archimedes' proof of this proposition is given above, pp. 456-7, and it is there pointed out that the result is equivalent to

12+22+32+ ...+n2=  n(n-1-1)(2n+1)

6

Colt. 1. *It follows from this proposition that*

*n•An2<3(Al2-1-A22+ • • •+An2),*

*and also that*

*n•An2>3(Al2-FA22+ • •* --FAn-12).

[For the proof of the latter inequality see p. 457 above.]

COR. 2. *All the results will equally hold if similar figures are substituted for*

*squares.*

PROPOSITION 11

*[in*

*If* **A1, A2, • • *'An*** *be n lines forming an ascending arithmetical progression which the common difference is equal to the least term Al], then*

*(n-1)An2 :(An2-FA,i2+ • • •+A22)<An2 :* {A.-A14-1(An—A1)2); *but*

*(n-1)An2 :(24,12+A.-22+ • • •-i-Al2)>An2 : {An •Ai-FEA.—A1)2}.* [Archimedes sets out the terms side by side in the manner shown in the figure, where *BC=An, DE= c . .RS= A1,* and produces *DE, FG, ...RS* until they are respectively equal to *BC* or *An,* so that *EH,*

T U

*GI, . SU* in the figure are respectively equal to *A1,*

**A2...** An\_1. He further measures lengths *BK, DL,*

*FM, ...PV* along *BC, DE, FG, ...PQ* respectively

each equal to *RS.*

The figure makes the relations between the terms

easier to see with the eye, but the use of so large a **K**- **L- M**

number of letters makes the proof somewhat difficult

to follow, and it may be more clearly represented as a **o**

follows.]

It is evident that *(An— A1)=*

The following proportion is therefore obviously true, viz.

(n-1)A„2 : *(n-1)(An•ArfiA.-12)=An2 : {An •A1-i-i(A.—A1)2}•*

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In order therefore to prove the desired result, we have only to show that (n- 1).A. •Ai+I(n- 1)An-12<(An2+An-12+ • • •±A22)

but > *(A* n-12±An-22+ • • •±Al2).

1. To prove the first inequality, we have

(n- 1)A,, •Al+I(n-1)An\_42

=(n-1)Al2+(n- *1)A1* •An-i+1(n-1)A....12. (1)

And

A.24-A.-12+ • • •-i-A = (An-i+A1)2+(An\_2+Ai)2+ • • -+(.A1-1--A1)2

=(A„..42+An\_22+ • • •+/1.12)

+(n-1)Al2

+2Ai(An-i+A.-2+ • • •+A1)

..---(A,,i2+An...22+ • • •+Al2)

+(n- 1),412

+AliA,i+An\_2+An\_a+ • • •+Ai

+Al+A2+ • • •+An-2+A„\_.1i

=(An\_12+A.-22+ • • •+Al2)

+(n-1)Al2

+n211 •An\_1. (2)

Comparing the right-hand sides of (1) and (2), we see that (n- 1)Al2 is com-

mon to both sides, and

(n-1)A1•An\_i<nAi•A,1,

while, by Prop. 10, Cur. 1,

i(n-1)A.-12<A,-12-1-A.-22+ • • -MI2.

It follows therefore that

(n-1)An .A1+•I(n-1)An-12<(An2.l..An\_12+ • • •+A22);

and hence the first part of the proposition is proved.

H. We have now, in order to prove the second result, to show that

(n- 1)A. •Ai+i(n- 1)A n\_12> *(A* n-12+A n\_22+ • • •+Al2).

The right-hand side is equal to

(A,,2+A1)2+ (An\_3+A1)2+ • • • + *(A* +Ai)2+A 12

:----An\_..22-1-An\_a2+ • • •+Ai2

+(n-1)Al2

* 2A1(An-2+An-3+ • • • +Ai)  
  = *(A* n-22+An-32+ • • •+Al2)
* (n- 1)Al2

+Alf A.-2+An-s+ • • •+A1

1 +AI +A2 + • • •+A.....2f

=(An\_22+An\_32+ • • .+Al2)

+(n-1)Al2

+ (n - 2)A I •An-1. (3)  
Comparing this expression with the right-hand side of (1) above, we see that (n-1)Al2 is common to both sides, and

(n- 1)A1 (n - 2)A 1

while, by Prop. 10, Cor. 1,

1(n— 1 )A,,\_12 > (A,22+An-32+ • • •+Al2)•

Hence (n- 1 )A „ *•A* + (n - 1)A n\_12> (An\_12+A.-22+ • • • +Ai2)

and the second required result follows.

Con. *The results in the above proposition are equally true if similar figures be*

*substituted for squares on the several lines.*

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DEFINITIONS

1. If a straight line drawn in a plane revolve at a uniform rate about one extremity which remains fixed and return to the position from which it started, and if, at the same time as the line revolves, a point move at a uniform rate along the straight line beginning from the extremity which remains fixed, the point will describe a *spiral* (04 in the plane.
2. Let the extremity of the straight line which remains fixed while the straight line revolves be called the *origin* of the spiral.
3. And let the position of the line from which the straight line began to revolve be called the *initial line* in the revolution.
4. Let the length which the point that moves along the straight line de­scribes in one revolution be called the *first distance,* that which the same point describes in the second revolution the *second distance,* and similarly let the distances described in further revolutions be called after the number of the particular revolution.
5. Let the area bounded by the spiral described in the first revolution and the *first distance* be called the *first* area, that bounded by the spiral described in the second revolution and the *second distance* the *second area,* and similarly for the rest in order.
6. If from the origin of the spiral any straight line be drawn, let that side of it which is in the same direction as that of the revolution be called *forward (rpoa7oiweva),* and that which is in the other direction *backward (i7r6Aeva.).*
7. Let the circle drawn with the *origin* as centre and the *first distance* as radius be called the *first circle,* that drawn with the same centre and twice the radius the *second circle,* and similarly for the succeeding circles.

**PROPOSITION** 12

*If any number of straight lines drawn from the origin to meet the spiral make equal*

*angles with one another, the lines will be in arithmetical progression.*

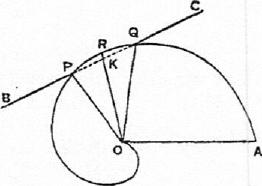
[The proof is obvious.]

**PROPOSITION** 13

*If a straight line touch the spiral, it will touch it in one point only.*

Let 0 be the origin of the spiral, and *BC* a tangent to it.

If possible, let *BC* touch the spiral in two



points *P, Q.* Join *OP, OQ,* and bisect the angle

*POQ* by the straight line *OR* meeting the

spiral in *R.*

Then [Prop. 12] *OR* is an arithmetic mean

between *OP* and *OQ,* or

*OP-FOQ=20R.*

But in any triangle *POQ,* if the bisector of the

angle *POQ* meets *PQ* in *K,*

*OP-FOQ>20K.*

Therefore *OK <OR,* and it follows that some point on *BC* between *P* and *Q*

lies within the spiral. Hence *BC* cuts the spiral; which is contrary to the

hypothesis.

ON SPIRALS  
**PROPOSITION** 14

*If 0 be the origin,* and *P, Q two points on the first turn of the spiral,* and *if OP, OQ produced meet the "first circle" AKP'Q' in P',* Q' *respectively, OA being the initial line, then*

|  |  |  |
| --- | --- | --- |
|  | **A** | *OP :0Q= (arc AKP') : (arc AKQ').*  For, while the revolving line *OA* moves about *0,* the point *A* on it moves uniformly along the circumference of the circle *AKP'Q' ,* and at the same time the point describing the spiral moves uniformly along *OA.*  Thus, while *A* describes the arc *AKP',* the moving point on *OA* describes the length *OP,* and, while *A* describes the arc *A KQ',* the moving point on *OA* describes the distance *OQ.*  Hence  *OP : OQ=* (arc *AKP') :* (arc *AKQ').* [Prop. 2] |

**PROPOSITION** 15

*If P, Q be points on the second turn of the spiral, and OP, OQ meet the "first circle" AKP'Q' in P', Q', as in the last proposition, and if c be the circumference of the `first circle," then*

*OP : OQ* =c+(arc *AKP') : c+* (arc *AKQ').*

For, while the moving point on *OA* describes the distance *OP,* the point *A* describes the whole of the circumference of the "first circle" together with the arc *AKP';* and, while the moving point on *OA* describes the distance *OQ,* the point *A* describes the:whole circumference of the "first circle" together with the arc *AKQ'.*

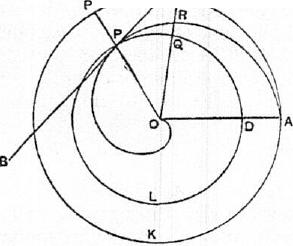
**COR.** Similarly, if *P, Q* are on the nth turn of the spiral,

*OP : OQ = (n 1)c+* (arc *AKP') : (n 1)c+* (arc *AKQ').*

**PROPOSITIONS** 16, 17

*If BC be the tangent at P, any point on the spiral, PC being the "forward" part of BC, and if OP be joined, the angle OPC is obtuse while the angle OPB is acute.*

c I. Suppose *P* to be on the first turn of  
the spiral.



Let *OA* be the initial line, *AKP'* the "first circle." Draw the circle *DLP* with centre *0* and radius *OP,* meeting *OA* in D. This circle must then, in the "forward" direction from *P,* fall within the spiral, and in the "backward" direction outside it, since the radii vectores of the spiral are on the "forward side" greater, and on the "backward" side less, than *OP.* Hence the angle *OPC* cannot be acute, si:ace it can­not be less than the angle between *OP* and

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the tangent to the circle at *P,* which is a right angle.

It only remains therefore to prove that *OPC* is not a right angle.

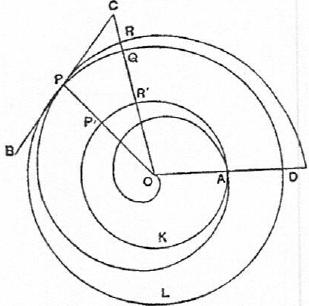
If possible, let it be a right angle. *BC* will then touch the circle at *P.*

Therefore [Prop. 5] it is possible to draw a line *OQC* meeting the circle

through *P* in *Q* and *BC* in *C,* such that

*CQ : OQ <* (arc *PQ) :* (arc *DLP).* (1)  
Suppose that *OC* meets the spiral *in R* and the "first circle" in *R' ;* and produce *OP* to meet the "first circle" in *P'.*

From (1) it follows, *componendo,* that *CO* : *OQ <* (arc *DLQ) :* (arc *DLP)*



< (arc *AKR') :* (arc *AKP')*

*<OR : OP.* [Prop. 14] But this is impossible, because *0Q=OP,* and *OR <OC.*

Hence the angle *OPC* is not a right angle. It was also proved not to be acute.

Therefore the angle *OPC* is obtuse, and the angle *OPB* consequently acute.

II. If *P* is on the second, or the nth turn, the proof is the same, except that in the proportion (1) above we have to substitute for the arc *DLP* an arc equal to (p+arc *DLP)* or (n-1 •p-1-arc *DLP),* where p is the perimeter of the circle *DLP* through *P.* Similarly, in the later steps, p or (n— 1)p will be added to each of the arcs *DLQ* and *DLP,* and c or (n —1)c to each of the arcs *AKR', AKP',* where *c* is the circumference of the "first circle" *AKP'.*

PROPOSITIONS 18, 19

*I. If OA be the initial line, A the end of the first turn of ,the spiral, and if the tangent to the spiral at A be drawn, the straight line OB drawn from 0 perpendic­ular to OA will meet the said tangent in some point B, and OB will be equal to the circumference of the "first circle."*

1. *If A' be the end of the second turn, the perpendicular OB will meet the tan­gent at A' in some point B', and OB' will be equal to* 2 *(circumference of "second circle").*
2. *Generally, if An be the end of the nth turn, and OB meet the tangent at*

***An*** *in* **Bn,** *then OBn= nen,*

*where c„ is the circumference of the "nth circle."*

I. Let *AKC* be the "first circle." Then, since the "backward" angle between *OA* and the tangent at *A* is acute [Prop. 16], the tangent will meet the "first circle" in a second point *C.* And the angles *CAO, BOA* are together less than two right angles; therefore *OB* will meet *AC* produced in some point *B.*

Then, if *c* be the circumference of the first circle, we have to prove that

*OB=c.*

If not, *OB* must be either greater or less than *c.*

(1) If possible, suppose *OB> c.*

Measure along *OB* a length *OD* less than *OB* but greater than *c.*

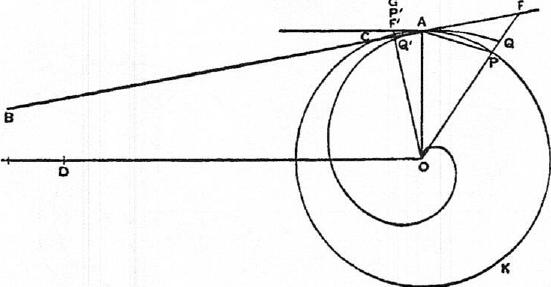
We have then a circle *AKC,* a chord *AC in* it less than the diameter, and a

ratio *AO : OD* which is greater than the ratio *AO : 013* or (what is, by similar

ON SPIRALS 91

triangles, equal to it) the ratio of *iAC* to the perpendicular from 0 on *AC.* Therefore [Prop. 7] we can draw a straight line *OPF,* meeting the circle in *P* and *CA* produced in *F,* such that

*FP : PA = AO :OD.*



**P'**

Thus, alternately, since *AO = P0,*

*FP : PO= PA : OD*

< (arc PA) : *c,*

since (arc *PA) > PA ,* and *OD > c.*

*Componendo,*

*FO : PO <(c-1-* arc *PA) : c*

*<OQ :OA,*

where *OF* meets the spiral in Q. [Prop. 15]  
Therefore, since OA *=OP, FO <OQ;* which is impossible.

Hence *OB>c.*

(2) If possible, suppose *OB <c.*

Measure *OE* along *OB* so that *OE* is greater than *OB* but less than *c.*

In this case, since the ratio *AO : OE* is less than the ratio *AO : OB* (or the

ratio of 1AC to the perpendicular from *0* on *AC),* we can [Prop. 8] draw a line

*OF'P'G,* meeting *AC* in *F',* the circle in *P',* and the tangent at *A* to the circle

in *G,* such that

*F'P' : AG =AO :OE.*

Let *OP'G* cut the spiral in *Q'.*

Then we have, alternately,

*F'P' : P'0= AG :OE*

> (arc *AP') : c,*

because *AG>* (arc *AP'),* and *OE <c.*

Therefore

*F'0 : P'0 <* (arc *AKP') : c*

*<OQ' :OA.* [Prop. 14]  
But this is impossible, since OA = *OP',* and *OQ' <OF'.*

Hence *OB*

Since therefore *OB* is neither greater nor less than *c,*

*OB =c.*

II. Let *A' K'C'* be the "second circle," *A'C'* being the tangent to the spiral

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at *A'* (which will cut the second circle, since the "backward" angle *OA'C'* is acute). Thus, as before, the perpendicular *OB'* to OA' will meet *A'C'* produced in some point *B'.*

If then *c'* is the circumference of the "second circle," we have to prove that

*OB' =* 2c'.

|  |  |
| --- | --- |
| **E'** |  |

For, if not, *OB'* must be either greater or less than 2c'.

(1.) If possible, suppose *OB' >2c`.*

Measure OD' along *OB'* so that OD' is less than *OB'* but greater than 2c'.

Then, as in the case of the "first circle" above, we can draw a straight line

*OPF* meeting the "second circle" in *P* and C'A' produced in *F,* such that

*FP : PA' = A'0 :OD'.*

Let *OF* meet the spiral in Q.

We now have, since *A'0 = PO,*

*PP : PO = PA' :OD'*

< (arc *A'P) :* 2c',

because (arc *A' P)> A'P* and *OD' >* 2c'.

Therefore *FO : PO <* (2c' + arc *A'P) :* 2c'

*<OQ : OA'.* [Prop. 15, Cor.]  
Hence *FO <OQ;* which is impossible.

Thus *OB' >* 2c'.

Similarly, as in the case of the "first circle," we can prove that

*OB' -4:* 2c'.

Therefore *OB' = 2c'.*

III. Proceeding, in like manner, to the "third" and succeeding circles, we

shall prove that

*OB.* **= *ncn.***

**PROPOSITION** 20

1. *If P be* any *point on the first turn of the spiral and OT be drawn perpendic­ular to OP, OT will meet the tangent at* ***P*** *to the spiral in some point T; and, if the circle drawn with centre 0 and radius OP meet the initial line in K, then OT is equal to the arc of this circle between K and P measured in the "forward" direction of the spiral.*

ON SPIRALS 93

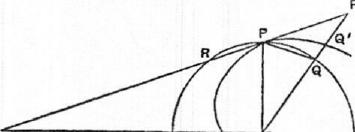
II. *Generally, if P be a point on the nth turn, and the notation be as before,*

*while p represents the circumference of the circle with radius OP,*

*OT = (n—* 1)p+ *arc KP (measured "forward").*

1. Let *P* be a point on the first turn of the spiral, *OA* **the** initial line, *PR* the tangent at *P* taken in the "backward" direction.

Then [Prop. 16] the angle *OPR is* acute. Therefore *PR* meets the circle through *P* in some point *R;* and also *OT* will meet *PR* produced in some point *T.*



If now *OT* is not equal to the arc *KRP,* it must be either great­er or less.

(1) If possible, let *OT* be greater than the arc *KRP.*

Measure *0* Ualong *OT* less than *OT* but greater than the arc *KRP.*

Then, since the ratio *PO : OU*

**A** is greater than the ratio *PO : OT,*or (what is, by similar triangles, equal to it) the ratio of *IPR* to the perpendicular from *0* on *PR,* we can draw a line *OQF,* meeting the circle in Q and *RP* produced in *F,* such that

*FQ : PQ=P0 :OU.* [Prop. 7]

Let *OF* meet the spiral in *Q'.*

We have then

*FQ :Q0=PQ :OU*

< (arc *PQ) :* (arc *KRP),* by hypothesis.

*Componendo,*

*FO : QO <* (arc *KRQ) :* (arc *KRP)*

*<OQ' : OP. [Prop.* **14]**

**But** *QO =OP.*Therefore *FO <OQ';* which is impossible.

Hence *OT>* (arc *KRP).*

(2) The proof that *OT <* (arc *KRP)* follows the method of Prop. 18, I. (2),

exactly as the above follows that of Prop. 18, I. (1).

Since then *OT* is neither greater nor less than the arc *KRP,* it is equal to it.

1. If *P* be on the second turn, the same method shows that

*OT =p+* (arc *KRP);*

and, similarly, we have, for a point *P* on the nth turn,

*OT = (n —1)p+* (arc *KRP).*

**PROPOSITIONS** 21, 22, 23

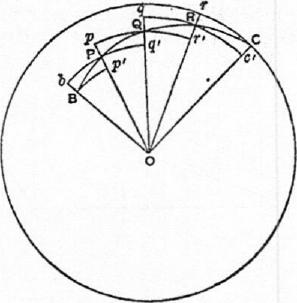
*Given* an *area bounded by any arc of a spiral and the lines joining the extremities of the arc to the origin, it is possible to circumscribe about the area one figure, and to inscribe in it another figure, each consisting of similar sectors of circles, and such that the circumscribed figure exceeds the inscribed by less than any assigned area.*

For let *BC* be any arc of the spiral, *0* the origin. Draw **the circle with centre**

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*0* and radius *OC,* where *C* is the "forward" end of the arc.

Then, by bisecting the angle *BOC,* bisecting the resulting angles, and so on continually, we shall ultimately arrive at an angle *COr* cutting off a sector of the circle less than any assigned area. Let *COr* be this sector.



Let the other lines dividing the angle *BOC* into equal parts meet the spiral in *P, Q,* and let *Or* meet it in *R.* With 0 as centre and radii *OB, OP, OQ, OR* respec­tively describe arcs of circles *Bp', bBq' , pQr', qRc' ,* each meeting the adjacent radii as shown in the figure. In each case the arc in the "forward" direction from each point will fall within, and the arc in the "backward" direction outside, the spiral.

We have now a circumscribed figure and an inscribed figure each consisting of similar sectors of circles. To compare their areas, we take the successive sectors of each, beginning from *OC,* and compare them.

The sector *OCr* in the circumscribed figure stands alone.

And (sector *ORq) =* (sector *()Re),*

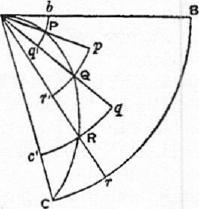
(sector *0Qp) =* (sector *0Qr'),*

(sector *OPb) =* (sector *OPq'),*

while the sector *0Bp'* in the inscribed figure stands alone.

Hence, if the equal sectors be taken away, the difference between the cir­cumscribed and inscribed figures is equal to the difference between the sectors *OCr* and *0Bp',* and this difference is less than the sector *OCr,* which is itself less than any assigned area.

The proof is exactly the same whatever be the number of angles into which the angle *BOC* is divided, the only difference being that, when the arc begins from the origin, the smallest sectors *OPb, OPq'* in each figure are equal, and there is therefore no inscribed sector standing by itself, so that the difference between the cir­cumscribed and inscribed figures is equal to the sector *OCr* itself.



Thus the proposition is universally true.

**COR.** Since the area bounded by the spiral is intermediate in magnitude between the circum­scribed and inscribed figures, it follows that

1. *a figure can be circumscribed to the area such that it exceeds the area by less than any assigned space,*
2. *a figure can be inscribed such that the area exceeds it by less than any assigned space.*

**PROPOSITION 24**

*The area bounded by the first turn of the spiral and the initial line is equal to one-third of the "first circle"* [=.1-7r(27ra)2, where the spiral is r = *a0].*

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[The same proof shows equally that, *if OP* be any *radius vector in the first turn of the spiral, the area of the portion of the spiral bounded thereby is equal to one-third of that sector of the circle drawn with radius OP which is bounded by the initial line and OP, measured in the "forward" direction from the initial line.]*

Let *0* be the origin, *OA* the initial line, *A* the extremity of the first turn.

Draw the "first circle," i.e. the circle with 0 as centre and *OA* as radius.

Then, if C1 be the area of the first circle, R1 that of the first turn of the spiral bounded by *OA,* we have to prove that

Ri = lei.

For, if not, R1 must be either greater or less than C1.

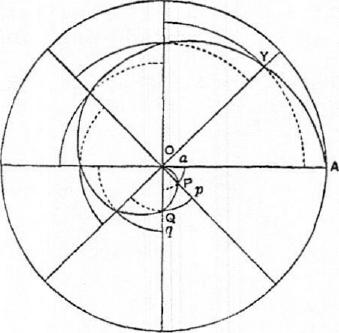
1. If possible, suppose RI <

We can then circumscribe a figure about R1 made up of similar sectors of

circles such that, if *F* be the area of this figure,

whence *F*

Let *OP, 0Q, • • •* be the radii of the circular sectors, beginning from the smallest. The radius of the largest is of course *OA.*



The radii then form an ascending arithmetical progression in which the common difference is equal to the least term *OP.* If n be the number of the sectors, we have [by Prop. 10, Cor. 1]

n • 0A2<3(0/32+0Q2-1- • • • *+OLP);* and, since the similar sectors are pro­portional to the squares on their radii, it follows that

C1< *3F,*

or *F>*

But this is impossible, since *F* was less than 1C1.

Therefore R1<-1-C1.

1. If possible, suppose RI> WI.

We can then *inscribe* a figure made up of similar sectors of circles such that,

if *f* be its area,



whence *f>*

If there are (n-1) sectors, their radii, as *OP, OQ, • •* form an ascending

arithmetical progression in which the least term is equal to the common differ-

ence, and the greatest term, as *OY,* is equal to (n— *1)0P.*

Thus [Prop. 10, Cor. 1]

n -0A2>3(0P2-1-0Q2-1- • • •-1-OY2),

whence C1 > 3f,

or *f*

which is impossible, since *f*

Therefore *R1>* -4C1.

Since then RI is neither greater nor less than iC1,

RI =WI.

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PRoPosirrioNs 25, 26, 27

[Prop. 25.] *If A2 be the end of the second turn of the spiral, the area bounded by the second turn and OA* **2 */8*** *to the area of the "second circle" in the ratio of 7 to* 12, *being the ratio of* {r2rid-i(r2—r02) *to* r22, *where r1, r2 are the radii of the "first" and "second" circles respectively.*

[Prop. 26.] *If BC be any arc measured in the "forward" direction on any turn of a spiral, not being greater than the complete turn, and if a circle be drawn with 0 as centre and OC as radius meeting OR in B', then*

*(area of spiral between OB, OC) : (sector OB'C)*

= *{0C • 0B+ROC —0B)2} : 0C2.*

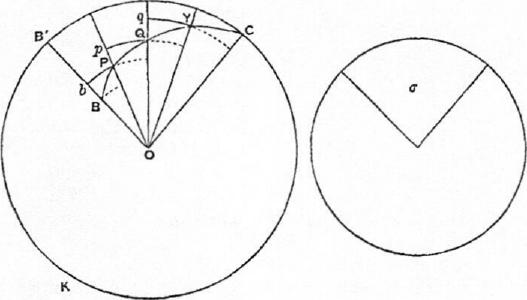
[Prop. 27.] *If R1 be the area of the first turn of the spiral bounded by the initial line,* **R2** *the area of the ring added by the second complete turn,* **R3** *that of the ring added by the third turn, and so on, then*

*R3=2R2, R4=3R2, R5= 4R2, • . Rn= (n-1)R2.*

*Also* **R2 =**

[Archimedes' proof of Prop. 25 is, *mutatis mutandis,* the same as his proof of the more general Prop. 26. The latter will accordingly be given here, and applied to Prop. 25 as a particular case.]

Let *BC* be an arc measured in the "forward" direction on any turn of the spiral, *CKB'* the circle drawn with 0 as centre and *OC* as radius.



Take a circle such that the square of its radius is equal to

*OC* • *0B+1(0C-0B)2,*

and let ***a*** he a sector in it whose central angle is equal to the angle *BOC.*

Thus ***a :*** (sector *OB'C) = OC* • *OB +1(OC —0B)21 :* 0C2,

and we have therefore to prove that

(area of spiral *OBC) = a.*

For, if not, the area of the spiral *ORC* (which we will call *S)* must be either

greater or less than ***a.***

***I.*** Suppose, if possible, ***S <a.***

Circumscribe to the area *S* a figure made up of similar sectors of circles, such

that, if *F* be the area of the figure,

***F 8<a —S,***

whence ***F <a.***

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Let the radii of the successive sectors, starting from *08,* be *OP, OQ, • • .0C.* Produce *OP, OQ, • • •* to meet the circle *CKB', • • •*

If then the lines *OB, OP, OQ,. . .0C* be n in number, the number of sectors in the circumscribed figure will be (n-1), and the sector *OB'C* will also be

divided into (n-1) equal sectors. Also *OB, OP, OQ, • • •OC* form an ascend-

ing arithmetical progression of n terms.

Therefore [see Prop. 11 and Cor.]

*(n-1)0C2 (0p2+0(22+ • • .+0C2) <0C2 IOC* • *0B+1(0C —0B)21*

< (sector *OB'C) :* ***a,*** by hypothesis.

Hence, since similar sectors are as the squares of their radii,

(sector *OB'C) : F <* (sector *OB'C)* : *cr,*

so that *F>* ***a.***But this is impossible, because

Therefore 8<a.

II. Suppose, if possible, *S> 0-.*

Inscribe in the area *S* a figure made up of similar sectors of circles such that,

if f be its area,

*S—f<S—o-,*

whence *f> o-.*

Suppose *OR, OP,. . .0Y* to be the radii of the successive sectors making up

the figure *f,* being (n-1) in number.

We shall have in this case [see Prop. 11 and Cor.]

(n 1)0C2 : *(0B2+0P2+* • • • + Or) > 0C2 : 10C • *0B-4-1(0C —0B)21,*

whence (sector *OB'C) : f>* (sector *OB'C) :* ***a,***

so that *f <a•*

But this is impossible, because *f>* ***a.***

Therefore S> a.

Since then *S* is neither greater nor less than Cr, it follows that

*S=cr.*

In the particular case where *B* coincides with A1, the end of the first turn of

the spiral, and *C* with *A* **2,** the end of the second turn, the sector *013'C* becomes

the complete "second circle," that, namely, with *0A2* (or r2) as radius.

Thus (area of spiral bounded by *0A2) :* ("second circle")

= {r2ri-Fi(r2—r02} : r22

= (2+1) : 4 (since r2= 21%)

=7 : 12.

Again, the area of the spiral bounded by *0A2* is equal to Ri+ R2 (i.e. the area

bounded by the first turn and *OA',* together with the ring added by the second

turn). Also the "second circle" is four times the "first circle," and therefore

equal to 12 R1.

Hence (R1+R2) : 12R1= 7 : 12,

or Ri+R2=7Ri.

Thus **R2 = 681. (1)**

Next, for the third turn, we have

(RI+ **R2+ R3) :** ("third circle") = {r3 r2+/(r3—r2)21 : r32

= (3 • 2+1) : 32

=19 : 27,

and ("third circle") =9("first circle")

=27R1;

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therefore Ri+R2+R3=19R1,  
and, by (1) above, it follows that

**R3= 12Ri**

= 282, **(2)**

and so on.

Generally, we have

(R1+R2+ • • • *+Rh) :* (nth circle) = try, —rn\_i)21 **rn2,  
(RIA-R2+ • • • +Rn-1) :** *(n—* lth circle) = frn\_1 rn\_2+Ern\_i—r.-2r1 :

and (nth circle) : (n— lth circle) =r7,2 :  
Therefore

*(R1+R2+ • • •-FR.) : (Ri-ER2+ • • • -1-R.-1)* = tn(n-1)+11 : f (n-1)(n-2)+-11

= f3n(n-1)+11 : 13(n-1)(n-2)+11.

*Dirimendo,*

**Rn : (RI +R2+ • • •** -FIL..1) =6(n — 1) : 13(n — 1)(n —2) +11. (a)

Similarly

: (Ri+R2+ • • --ER.--2) =6(n-2) : 13(n-2)(n-3)+11,

from which we derive

: (Ri+R2+ • • •+Rn-i)

=6(n-2) : f 6(n-2)+3(n-2)(n-3)+11

=6(n — 2) : f 3(n — 1)(n — 2) + 11. *(13).*Combining (a) and *((3),* we obtain

**Rn :** (n-1) : (n-2).

Thus

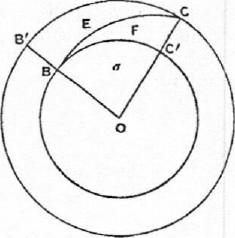
R2, **R3, R4, • •** are in the ratio of the successive numbers 1, 2, 3 • • • *(n-1).*

**PROPOSITION** 28

*If 0 be the origin and BC any arc measured in the "forward" direction on any turn of the spiral, let two circles be drawn (1) with centre 0, and radius OB, meeting OC in C', and* (2) *with centre 0 and radius OC, meeting OB produced in B'. Then, if E denote the area bounded by the larger circular arc B'C, the line B'B, and the spiral BC, while F denotes the area bounded by the smaller arc BC', the line CC' and the spiral BC,*

*E : F= {0B-11(0C —0B)} :* **{** *0/3±1-(0C —0B)}.*

Let ***o*** denote the area of the lesser sector *OBC' ;*



then the larger sector *OB'C* is equal to ***cr +F +E.***

Thus [Prop. 26]

***(o+F) :*** *(a+F+E)=*

*{OC* • *0B+1(0C-0B)2}* : 0C2, (1)

whence

*E : (0-4-F) = {0C(OC — OB) —1(0C — OB)2}*

: {*0C • 0B+ROC—OB)21*

*= {0B(OC OB) +ROC —0B)2}*

*: {OC •* ***OB +(OC —0B)2}.*** (2)

Again

***F F E) : o-=0C2 : OB2.***

Therefore, by the first proportion above, *ex aequali,*

***(o- F) : o=*** *{OC* • *0B+-1,-(0C-0B)21* : *OB2,*

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whence

*(a + F) : F= lOC • 0B+EOC —0B)2}*

: f *OB(OC* — *OB)±* i *(0C— OB)21* .

Combining this with (2) above, we obtain

*E : F= 10B(OC —0B)+ -3(0C —0B)21 : {0B(OC —0B)+13.(0C —0B)21*

*= 10B-1-1(0C —0B)1 : tOB +1(0C —0B)).*

**ON THE EQUILIBRIUM OF PLANES OR  
THE CENTRES OF GRAVITY OF PLANES**

**BOOK ONE**

**"1 POSTULATE** the following":

1. "Equal weights at equal distances are in equilibrium, and equal weights at unequal distances are not in equilibrium but incline towards the weight which is at the greater distance."
2. "If, when weights at certain distances are in equilibrium, something be added to one of the weights, they are not in equilibrium but incline towards that weight to which the addition was made."
3. "Similarly, if anything be taken away from one of the weights, they are not in equilibrium but incline towards the weight from which nothing was taken."
4. "When equal and similar plane figures coincide if applied to one another, their centres of gravity similarly coincide."
5. "In figures which are unequal but similar, the centres of gravity will be similarly situated. By points similarly situated in relation to similar figures I mean points such that, if straight lines be drawn from them to the equal angles, they make equal angles with the corresponding sides."
6. "If magnitudes at certain distances be in equilibrium, (other) magnitudes equal to them will also be in equilibrium at the same distances."
7. "In any figure whose perimeter is concave in (one and) the same direction the centre of gravity must be within the figure."

**PROPOSITION 1**

*Weights which balance at equal distances are equal.*

For, if they are unequal, take away from the greater the difference between the two. The remainders will then not balance *[Post.* 3] ; which is absurd. Therefore the weights cannot be unequal.

**PROPOSITION 2**

*Unequal weights at equal distances will not balance but will incline towards the greater weight.*

For take away from the greater the difference between the two. The equal remainders will therefore balance *[Post.* 1]. Hence, if we add the difference again, the weights will not balance but incline towards the greater *[Post.* **2].**

**PROPOSITION** 3

*Unequal weights will balance at unequal distances, the greater weight being at the*

*lesser distance.*

Let *A, B* be two unequal weights (of which *A* is the greater) balancing about

*C* at distances *AC, BC* respectively.

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ON THE EQUILIBRI•UM OF PLANES I 101

Then shall *AC* be less than *BC.* For, if not, take away from *A* the weight (A — *B).* The remainders will then incline towards *B [Post.* 3]. But this is impossible, for (1) if A *C. CB,* the equal remainders will balance, or (2) if *AC>CB,*

they will' incline towards *A* at the

greater distance *[Post. 1].*

Hence *AC <CB.*

*Conversely,* if the weights balance, and *AC <CB,* then *A> B.*

PROPOSITION 4

*If two equal weights have not the same centre of gravity, the centre of gravity of both  
taken together is at the middle point of the line joining their* centres *of gravity.*[Proved froln Prop. 3 by *reductio ad absurdum.]*

PROPOSITION 5

*If three equal magnitudes have their centres of gravity on a straight line at equal distances, the centre of gravity of the system will coincide with that of the middle magnitude.*

[This follows 'immediately from Prop. 4.]

COR. 1. *Thesarne is true of arty odd number of magnitudes if those which are at equal. distances from the middle one* are *equal, while the distances between their centres of gravity are equal.*

COR. 2. *If there be an even number of magnitudes with their centres of gravity situated at equal distances on one straight line, and if the two middle ones be equal, while those which are equidistant from them (on each side) are equal respectively, the centre of gravity of the system is the middle point of the line joining the centres of gravity of the two middle ones.*

PROPOSITIONS 6, *7*

*Two magnitudes, whether commensurable* [Prop. 6] *or incommensurable* [Prop. 7], *balance at distances reciprocally proportional to the magnitudes.*

I. Suppose the magnitudes *A, B* to be commensurable, and the points *A, B* to be their centres of gravity. Let *DE* be a straight line so divided at *C* that *A : B =DC :CE.*

We have then to prove that, if A be placed at *E* and *B* at *D, C* is the centre of gravity of the two taken

o together.

Since *A.,13* are commensurable, so are *DC, CE.* Let *N* be a common measure of *DC, CE.* Make *DH, DK*

N

is bisected at *E, as HK* is bisected at *D.*

Thus *LH, HK* must each contain N an even number of times.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **A** |  | a |  |  |
|  |  |  |
|  |  |  |
|  |  |  |  |

c

**H**

i I D

**K** each equal to *CE,* and *EL* (on *CE*

produced) equal to *CD.* Then *EH = CD,* since *DII =CE.* Therefore *LH*

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**Take a magnitude 0 such that *0* is contained as many times in *A* as *N* is**

**contained in *LH,* whence**

**.1 : 0 = *LH : N.***

**But *B : A =CE : DC***

***=HK :LH.***

**Hence, *ex aequali, B : 0 =11K : N,* or *0* is contained in *B as* many times *as N* is contained in *HK.***

**Thus *0* is a common measure of *A, B.***

**Divide *LH, HK* into parts each equal to *N,* and *A, B* into parts each equal to 0. The parts of A will therefore be equal in number to those of *LH,* and the parts of *B* equal in number to those of *HK.* Place one of the parts of *A* at the middle point of each of the parts *N* of *LH,* and one of the parts of *B* at the middle point of each of the parts *N* of *HK.***

**Then the centre of gravity of the parts of *A* placed at equal distances on *LH* will be *at E,* the raiddle.point of *LH* [Prop. 5, Cor. 2], and the centre of gravity of the parts of *B* placed at equal distances along *HK* will be at *D,* the middle point of *11K.***

**Thus we may suppose *A* itself applied at *E,* and *B* itself applied at D.**

**But the system formed by the parts 0 of *A* and *B* together is a system of equal magnitudes even in number and placed at equal distances along *LK.* And, since *LE = CD,* and *EC = DK, LC = CK,* so that *C* is the middle point of *LK.* Therefore *C* is the centre of gravity of the system ranged along *LK.***

**Therefore *A* acting at *E* and *B* acting *at D* balance about the point *C.***

**II. Suppose the magnitudes to be incommensurable, and let them be *(A* +a) and *B* respectively. Let *DE* be a line divided at *C* so that**

***(A+a) : B=DC :CE.***

**Then, if (A +a) placed at *E* and**

***B* placed at *D* do not balance**

**about *C, (A + a)* is either too great to balance *B,* or not great enough.**

a **A**

**Suppose, if possible, that *(A +a)* is too great. to balance *B.* Take from *(A +a)* a magnitude a smaller than the deduction which would make the remainder balance *B,* but such that the remainder A and the magnitude *B* are commensurable.**

B

**Then, since *A, B* are commensurable, and**

**A : *B <DC :CE,***

***A* and *B* will not balance [Prop. 6], but *D* will be depressed.**

**But this is impossible, since the deduction a was an insufficient deduction from *(A +a)* to produce equilibrium, so that *E* was still depressed.**

**Therefore (A +a) is not too great to balance *B;* and similarly it may be proved that *B* is not too great to balance (A+a).**

**Hence *(A +a), B* taken together have their centre of gravity at *C.***

**PROPOSITION 8**

***If AB be a magnitude whose centre of gravity is C, and AD a part of it whose centre of gravity is F, then the centre of gravity of the remaining part will be a point G on FC produced such that***

ON THE EQUILIBRIUM OF PLANES **I** 103

*GC :CF= (AD) : (DE).*

For, if the centre of gravity of the re-

mainder *(DE)* be not *G,* let it be a point

*H.* Then an absurdity follows at once

from Props. 6, 7.

PROPOSITION 9

*The centre of gravity of any parallelogram lies on the straight line joining the middle points of opposite sides.*

Let *ABCD* be a parallelogram, and let *EF* join the middle points of the opposite sides *AD, BC.*

If the centre of gravity does not lie on *EF,* suppose it to be *H,* and draw *HK* parallel to *AD* or *BC* meeting *EF* in *K.*

Then it is possible, by bisecting *ED,* then bisecting the halves, and so on continually,

:////: to arrive at a length *EL* less than *KH.* Divide

both *AE* and *ED* into parts each equal to

*EL,* and through the points of division draw parallels to *AB* or *CD.*

We have then a number of equal and simi-

lar parallelograms, and, if any one be applied to any other, their centres of gravity coincide *[Post.* 4]. Thus we have an even number of equal magnitudes whose centres of gravity lie at equal distances along a straight line. Hence the centre of gravity of the whole parallelogram will lie on the line joining the centres of gravity of the two middle parallel­ograms [Prop. 5, Cor. 2].

But this is impossible, for *H* is outside the middle parallelograms. Therefore the centre of gravity cannot but lie on *EF.*

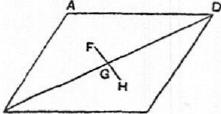
PROPOSITION 10

*The centre of gravity of a parallelogram is the point of intersection of its diagonals.* For, by the last proposition, the centre of gravity lies on each of the lines which bisect opposite sides. Therefore it is at the point of their intersection; and this is also the point of intersection of the diagonals.

*Alternative proof.*

Let *ABCD* be the given parallelogram, and *BD* a diagonal. Then the tri­angles *ABD, CDB* are equal and similar, so that *[Post.* 4], if one be applied to the other, their centres of gravity will fall one upon the other.

Suppose *F* to be the centre of gravity of the triangle *ABD.* Let *G* be the middle point of *BD.* Join *FG* and produce it to *H,* so that *PG = GH.*



If we then apply the triangle *ABD* to the tri­angle *CDB* so that *AD* falls on *CB* and *AB* **on** *CD,* the point *F* will fall on *H.*

But [by *Post.* 4] *F* will fall on the centre of grav-

ity of *CDB.* Therefore *H* is the centre of gravity of *CDB.*

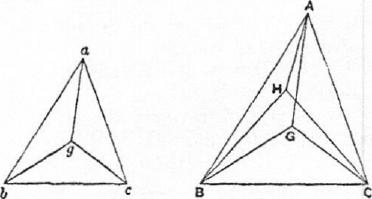
Hence, since *F, H* are the centres of gravity of the two equal triangles, the centre of gravity of the whole parallelogram is at the middle point of *FH,* i.e. at the middle point of *BD,* which is the intersection of the two diagonals.

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**PROPOSITION II**

*If abc, ABC be two similar triangles, and g, G two points in them similarly situ­ated with respect to them respectively, then, if g be the centre of gravity of the triangle abc, G must be the centre of gravity of the triangle ABC.*

Suppose



*ab :bc :ca=AB :BC :CA.*

The proposition is proved by an obvious *reductio ad absurdum.* For, if *G* be not the centre of gravity of the triangle *ABC, sup­pose* ***H*** to be its centre of gravity.

*Post.* 5 requires that *g,* ***H*** shall be similarly situated with respect to the triangles respec­tively; and this leads at once to the absurdity that the angles *HAB,* GAB are equal.

**PROPOSITION 12**

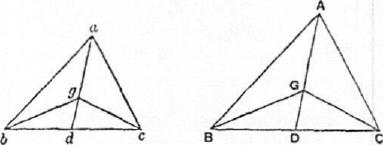
*Given two similar triangles abc, ABC, and d, D the middle* ***points of be, BC***

*respectively, then, if the centre of gravity of abc lie on ad, that of* ***ABC will*** *lie*

*on AD.*

Let *g* be the point on *ad* which

is the centre of gravity of *abc.*



Take *G* on ***AD*** such that

*ad : ag =AD : AG,*

and join *gb, gc, GB, GC.*

Then, since the triangles are

similar, and *bd, BD* are the

halves of *bc, BC* respectively,

*ab:bd=AB :BD,*

and the angles abd, *ABD* are equal.

Therefore the triangles *abd, ABD* are similar, and

*Lbad= L BAD.*

Also ba : *ad=BA :AD,*

while, from above, *ad : ag =AD : AG.*

Therefore ba : ag = *BA : AG,* while the angles *bag, BAG* are equal.

Hence the triangles *bag, BAG* are similar, and

*abg = L ABG.*

And, since the angles *abd, ABD* are equal, it follows that

***Lgbd= LGBD.***

In exactly the same manner we prove that

*L gac = LGAC,*

*Lacg= LACG,*

*.Cgcd= ZGCD.*

Therefore *g, G* are similarly situated with respect to the triangles respec-

tively; whence [Prop. 11] C, is the centre of gravity of *ABC.*

ON THE EQUILIBRIUM OF PLANES I 105

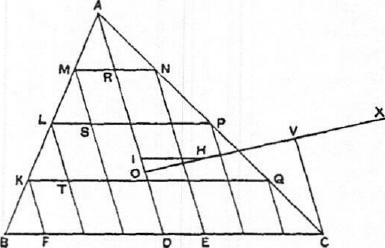
**PROPOSITION** 13

*In any triangle the centre of gravity lies on the straight line joining* any *angle to the middle point of the opposite side.*

Let *ABC* be a triangle and *D* the middle point of *BC.* Join *AD.* Then shall the centre of gravity lie on *AD.*

For, if possible, let this not be the case, and let *H* be the centre of gravity. Draw *HI* parallel to *CB* meeting *AD* in *I.*

Then, if we bisect *DC,* then bisect the halves, and so on, we shall at. length arrive at a length, as *DE,* less than *HI.* Divide both *BD* and *DC* into lengths each equal to *DE,* and through the points of division draw lines each parallel to *DA* meeting *BA* and *AC* in points as *K, L, M* and *N, P, Q* respectively.



Join *MN, LP, KQ,* which lines will then be each parallel to *BC.*

We have now a series of parallelograms as *FQ, TP, SN,* and *AD* bisects opposite sides in each. Thus the centre of gravity of each parallelogram lies on *AD* [Prop. 9], and therefore the centre of gravity of the figure made up of them all lies on *AD.*

Let the centre of gravity of all the parallelograms taken together be *0.* Join *OH* and produce it; also draw *CV* parallel to DA meeting *OH* produced in V. Now, if n be the number of parts into which *AC* is divided,

*AADC :* (sum of triangles on *AN, NP, • • •)= AC2 : (AN2+NP2+ • • .)*

*n***2***=***:** *n*

*n:1*

*=AC : AN.*

Similarly

*/ABD :* (sum of triangles on *AM, ML, • • •) =AB :AM.*

And *AC : AN=AB :AM.*

It follows that

*PABC* : (sum of all the small Ps) *= CA : AN*

*> VO : OH,* by parallels.

Suppose *OV* produced to *X* so that

*PA BC :* (sum of small Ps) = *XO* : *OH,*

whence, *dividendo,*

(sum of parallelograms) : (sum of small Ps) = *XH : HO.*

Since then the centre of gravity of the triangle *ABC* is at *H,* and the centre of

gravity of the part of it made up of the parallelograms is at 0, it follows from

Prop. 8 that the centre of gravity of the remaining portion consisting of all the

small triangles taken together is at *X.*

But this is impossible, since all the triangles are on one side of the line

through *X* parallel to *AD.*

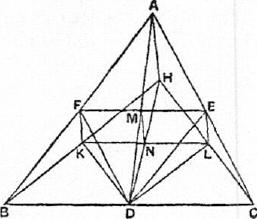
Therefore the centre of gravity of the triangle cannot but lie on *AD.*

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*Alternative proof.*

Suppose, if possible, that *H,* not lying on *AD,* is the centre of gravity of the triangle *ABC.* Join *All, BH, CH.* Let *E, F* be the middle points of *CA, AB* respectively, and join *DE, EF, FD.* Let *EF* meet *AD* in *M.*

Draw *FK, EL* parallel to *AH* meeting *BH, CH* in *K, L* respectively. Join *KD, HD, LD, KL.* Let *KL* meet *DH* in *N,* and join *MN.*



Since *DE* is parallel to *AB,* the triangles *ABC, EDC* are similar.

And, since *CE=EA,* and *EL* is parallel to *AH,* it follows that *CL = LH.* And *CD = DB.* Therefore *NI* is parallel to *DL.*

Thus in the similar and similarly situated triangles *ABC, EDC* the straight lines *AH, BH* are respectively parallel to *EL, DL;* and it follows that *H, L* are similarly situated with respect to the triangles respectively.

But *II* is, by hypothesis, the centre of gravity of *ABC.* Therefore L is the

centre of gravity of *EDC.* [Prop. 11]  
Similarly the point *K* is the centre of gravity of the triangle *FBD.*

And the triangles *FBD, EDC* are equal, so that the centre of gravity of both together is at the middle point of *KL,* i.e. at the point *N.*

The remainder of the triangle *ABC,* after the triangles *FBD, EDC* are de­ducted, is the parallelogram *AFDE,* and the centre of gravity of this paral­lelogram is at M, the intersection of its diagonals.

It follows that the centre of gravity of the whole triangle *ABC* must lie on *MN;* that *is, MN* must pass through *H,* which is impossible (since *MN* is parallel to *AH).*

Therefore the centre of gravity of the triangle *ABC* cannot but lie on *AD.*

PROPOSITION 14

It follows at once from the last proposition that *the centre of gravity of any triangle is at the intersection of the lines drawn from any two angles to the middle points of the opposite sides respectively.*

PROPOSITION 15

*If AD, BC be the two parallel sides of a trapezium ABCD, AD being the smaller,*

*and if AD, BC be bisected at E, F respectively, then the centre of gravity of the*

*trapezium is at a point G on EF such that*

*GE :GF (2BC-FAD) : (2AD-1-BC).*

Produce *BA, CD* to meet at *0.* Then *FE* produced Will also pass through *0,*

since *AE =ED,* and *BF*

Now the centre of gravity of the triangle *OAD* will lie on *OE,* and that of the

triangle *OBC* will lie on *OF.* [Prop. 13]  
It follows that the centre of gravity of the remainder, the trapezium *ABCD,*

will also lie on *OF.* [Prop. 8]  
Join *BD,* and divide it at *L, M* into three equal parts. Through *L, M* draw *PQ, RS* parallel to *BC,* meeting *BA* in *P, R, FE in W, V,* and *CD* in ***Q,.,S*** respectively,

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Hence [Props. 6, 73

*PDBC* : ***QABD= KG :GH***

***=VG :GIV.***

But ***/DBC : PABD=BC : AD.***

Therefore ***BC : A D = VG :GU'.***

It follows that

***(2BC- AD) : (2AD+ BC) = (2VG+GTV) : (2G1V+VG)***

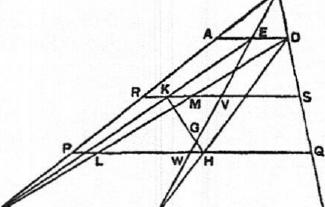
***=EG : GF.* Q.E.D.**

Join ***DF, BE*** meeting ***PQ*** in ***H*** and ***RS*** in ***K*** respectively.

Now, since ***BL =1BD,***

***FH.iFD.***

Therefore ***H is*** the centre of grav­ity of the triangle ***DBC.***



Similarly, since ***EK=1BE,*** it fol­lows that ***K*** is the centre of gravity of the triangle ***ADB.***

Therefore the centre of gravity of the triangles ***DBC, ADB*** together, i.e. of the trapezium, lies on the line ***HK.***

But it also lies on ***OF.***

Therefore, if ***OF, HK*** meet in ***G, G***

**9 r c** is the centre of gravity of the trape-

zium.

ON THE EQUILIBRIUM OF PLANES  
BOOK TWO

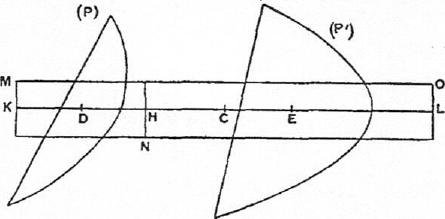
PROPOSITION 1

*If P, P' be two parabolic segments and D, E their centres of gravity respectively, the centre of gravity of the two segments taken together will be at a point C on DE determined by the relation*

*P :P'=CE :CD.*

In the same straight line with *DE* measure *Ell, EL* each equal to *DC,* 4nd *DK* equal to *DH;* whence it follows at once that *DK=CE,* and also that

*KC =CL.*



Apply a rectangle *MN* equal in area to the parabolic segment *P* to a base equal to *KH,* and place the rectangle so that *KH* bisects it, and is parallel to its base.

Then *D* is the centre of gravity of *MN,* since *KD = DH.*

Produce the sides of the rectangle which are parallel to *KH,* and complete the rectangle *NO* whose base is equal to *HL.* Then *E* is the centre of gravity of the rectangle *NO.*

*Now (MN) : (N0)=.1(1.1 : HL  
=DH :EH  
=CE :CT)  
=P : P'.*

But *(MN) = P.*

Therefore *(NO)=P'.*

Also, since *C* is the middle point of *KL, C* is the centre of gravity of the whole parallelogram made up of the two parallelograms *(MN), (NO),* which are equal to, and have the same centres of gravity as, *P, P'* respectively.

Hence *C* is the centre of gravity of *P, P'* taken together.

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DEFINITION AND LEMMAS PRELIMINARY TO PROPOSITION 2

"If in a segment bounded by a straight line and a section of a right-angled cone [a parabola] a triangle be inscribed having the same base as the segment and equal height, if again triangles be inscribed in the remaining segments having the same bases as the segments and equal height, and if in the remain­ing segments triangles be inscribed in the same manner, let the resulting figure be said to be *inscribed in the recognised manner* in the segment.

"And it is plain"

1. "that *the lines joining the two angles of the figure so inscribed which are near­est to the vertex of the segment, and the next pairs of angles in order, will be parallel to the base of the segment,"*
2. "that *the said lines will be bisected by the diameter of the segment,* and"
3. "that *they will cut the diameter in the proportions of the successive odd num­bers, the number one having reference to [the length adjacent to] the vertex of the segment.*

"And these properties will have to be proved in their proper places."

PROPOSITION 2

*If a figure be "inscribed in the recognised manner" in a parabolic segment, the centre of gravity of the figure so inscribed will lie on the diameter of the segment.*

For, in the figirre of the foregoing lemmas, the centre of gravity of the trapezium *BRrb* must lie on *XO,* that of the trapezium *RQqr* on *WX,* and so on, while the centre of gravity of the triangle *PAp* lies on *AV.*

Hence the centre of gravity of the whole figure lies on *AO.*

PROPOSITION 3

*If BAB', bab' be two similar parabolic segments whose diameters are AO, ao*

*respectively, and if a figure be inscribed in each segment "in the recognised man-*

*ner," the number of sides in each figure being equal, the centres of gravity of the*

*inscribed figures will divide AO, ao in the same ratio.1*

Suppose *BRQPAP'Q'R'B', brqpap'q'r'b'* to be the two figures inscribed "in

the recognised manner." Join *PP', QQ', RR'* meeting *AO* in L, *M, N,* and pp',

*qq', rr'* meeting *ao* in *1,* m, n.

Then [Lemma (3)]

*AL :LM :MN :NO=1 :* 3 : 5 : 7

*=al : lm : mn : no,*

so that *AO, ao* are divided in the same proportion.

Also, by reversing the proof of Lemma (3), we see that

*PP' : pp' =QQ' : qq' =RR' : rr' = BB' :* bb'.

Since then *RR' : BB'* = rr' : *bb',* and these ratios respectively determine the

proportion in which *NO, no* are divided by the centres of gravity of the tra-

pezia *B RR' B' , brr'b'* [r. 15], it follows that the centres of gravity of the trapezia

divide *NO, no* in the same ratio.

Similarly the centres of gravity of the trapezia *RQQ' R' , rqq'r'* divide *MN,*

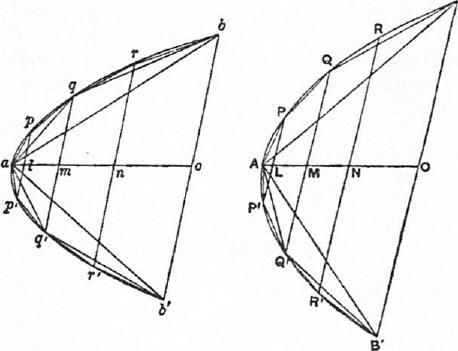
*inn* in the same ratio respectively, and so on.

lArehimedes enunciates this proposition as true of *similar* segments, but it is equally true of segments which are not similar, as the course of the proof will show.

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Lastly, the centres of gravity of the triangles *PAP', pap'* divide *AL, al* respectively in the same ratio.

**a**

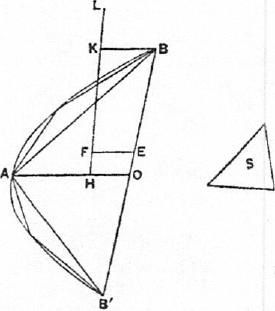


Moreover the corresponding trapezia and triangles are, each to each, in the same proportion (since their sides and heights are respectively proportional), while AO, *ao* are divided in the same proportion.

Therefore the centres of gravity of the complete inscribed figures divide *AO, ao* in the same proportion.

PROPOSITION 4

*The centre of gravity of any parabolic segment cut off by a straight line lies on the diameter of the segment.*



Let *BAB'* be a parabolic segment, A its vertex and *AO* its diameter.

Then, if the centre of gravity of the seg­ment does not lie on AO, suppose it to be, if possible, the point *F.* Draw *FE* parallel to AO meeting *BB'* in *E.*

Inscribe in the segment the triangle *ABB'* having the same vertex and height as the segment, and take an area *S* such that

*4!1,ABB' : 8=BE :BO.*

We can then inscribe in the segment "in the recognised manner" a figure such that the segments of the parabola left over are together less than *S.'*

1For Prop. 20 of the *Quadrature of the Parabola* proves that, if in any segment the triangle with the same base and height be inscribed, the triangle is greater than half the segment; whence it appears that, each time that we increase the number of the sides of the figure inscribed "in the recognised manner," we take away more than half of the remaining seg­ments.

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Let the inscribed figure be drawn accordingly; its centre of gravity then lies

on AO [Prop. 2]. Let it be the point *H.*

Join *HF* and produce it to meet in *K* the line through *B* parallel to *AO.*

Then we have

(inscribed figure) : (remainder of segmt.)> *L1 ABB' : S*

*>BE :EO*

*>KF : FH.*

Suppose *L* taken on *HK* produced so that the former ratio is equal to the ratio

*LF :PH.*

Then, since *H* is the centre of gravity of the inscribed figure, and F that of

the segment, L must be the centre of gravity of all the segments taken together

which form the remainder of the original segment. [I. 8]  
But this is impossible, since all these segments lie on one side of the line drawn through L parallel to *AO* (Cf. *Post.* 7].

Hence the centre of gravity of the segment cannot but lie on *AO.*

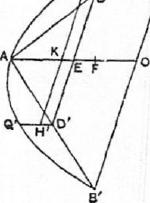
PROPOSITION 5

*If in a parabolic segment a figure be inscribed "in the recognised manner," the centre of gravity of the segment is nearer to the vertex of the segment than the centre of gravity of the inscribed figure is.*

Let *BAB'* be the given segment, and *AO* its diame­ter. *First,* let *ABB'* be the *triangle* inscribed "in the recognised manner."

Divide *AO* in *F* so that *AF = 2F0; F* is then the centre of gravity of the triangle *ABB'.*

Bisect *AB, AB'* in D, D' respectively, and join *DD'* meeting *AO* in *E.* Draw *DQ, D'Q'* parallel to *OA* to meet the curve. *QD, Q'D'* will then be the diameters of the segments whose bases are *AB, AB',* and the cen­tres of gravity of those segments will lie respectively on *QD, Q'D'* [Prop. 4]. Let them be *H, H',* and join *HIP* meeting *AO* in *K.*



Now *QD, Q'D'* are equal,' and therefore the seg­ments of which they are the diameters are equal *[On Conoids and Spheroids,* Prop. 3].

Also, since *QD, Q'D'* are parallel, and *DE = ED', K* is the middle point of

Hence the centre of gravity of the equal segments *AQB, AQ'B'* taken to­gether is *K,* where *K* lies between *E* and *A.* And the centre of gravity of the triangle *ABB' is F.*

It follows that the centre of gravity of the whole segment *BAB'* lies between *K* and *F,* and is therefore nearer to the vertex A than *F* is.

*Secondly,* take the *fire-sided* figure *BQAQ'B'* inscribed "in the recognised manner," *QD, Q'D'* being, as before, the diameters of the segments *AQB, AQ'B'.*

Then, by the first part of this proposition, the centre of gravity of the seg­ment *AQB* (lying of course on *QD)* is nearer to *Q* than the centre of gravity of

'This may either be inferred from Lemma (1) above (since *QQ', DD'* are both parallel to *BB'),* or from Prop. I of the *Quadrat re of the Parabola,* which applies equally to Q or V.

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the triangle *AQB* is. Let the centre of gravity of the segment be *//,* and that of

the triangle *I.*

Similarly let *II'* be the centre of gravity of the seg­ment *AQ'B',* and *I'* that of the triangle *AQ'B'.*

It follows that the centre of gravity of the two seg­ments *AQB, AQ'B'* taken together is *K,* the middle point of *1111',* and that of the two triangles *AQB, AQ'B'* is *L,* the middle point of *II'.*

If now the centre of gravity of the triangle *ABB'* be *F,* the centre of gravity of the whole segment *BAB'* (i.e. that of the triangle *ABB'* and the two segments *AQB, AQ'B'* taken together) is a point *G* on *1‘1,'* de­termined by the proportion

(sum of segments *AQB, AQ'B') : PABB' = FG : GK.*

[I. 6, 7]

And the centre of gravity of the inscribed figure

*BQAQ' B'* is a point *F'* on *LF* determined by the proportion

*(AAQB- AAQ'B') : AA BB' =FP : F'L.*

[Hence *FG : GK> FF' :*

or *GK :FG <F'L :FF',*

and, *componendo, FK : FG <FL : FF',* while *FK> FL.]* Therefore *FG> FF',* or *G* lies nearer than *F'* to the vertex *A .*

Using this last result, and proceeding in the same way, we can prove the proposition for any figure inscribed "in the recognised manner."

**PROPOSITION** 6

*Given a segment of a parabola cut off by a straight line, it is possible to inscribe in it "in the recognised manner" a figure such that the distance between the centres of gravity of the segment and of the inscribed figure is less than any assigned length.*

Let *BAB'* be the segment, *AO* its diam­eter, *G* its centre of gravity, and *ABB'* the triangle inscribed "in the recognised manner."

Let *D* be the assigned length and *S* an area such that

*AG : D= AABB' :S.*

In the segment inscribe "in the recog­nised manner" a figure such that the sum of the segments left over is less than *S.* Let *F* be the centre of gravity of the in­scribed figure.

We shall prove that *PG <D.*

For, if not, *FG* must be either equal to, or greater than, *D.*

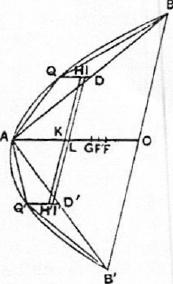
And clearly

(inscribed fig.) : (sum of remaining segmts.)

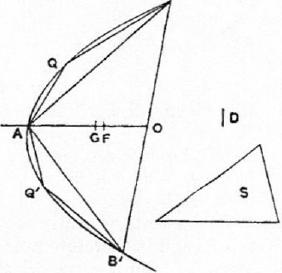
*>PABB'* : *S*

*>AG : D*

*> AG : FG, by* hypothesis (since *FG <D).*



[I. 6, 7]



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Let the first ratio be equal to the ratio *KG : FG* (where *K* lies on *GA* pro­duced); and it follows that *K* is the centre of gravity of the small segments

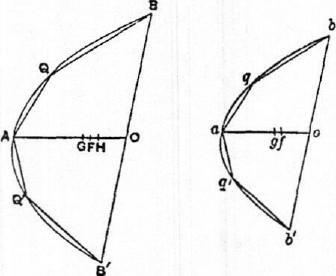
taken together. [I. 8]  
But this is impossible, since the segments are all on the same side of a line drawn through *K* parallel to *BB'.*

Hence *FG* cannot but be less than D.

**PROPOSITION** *7*

*If there be two similar parabolic segments, their centres of gravity divide their diameters in. the same ratio.*

Let *BAB', bab'* be the two similar segments, AO, *ao* their diameters, and *G, g* their centres of gravity re­spectively.



Then, if *G, g* do not divide *AO, ao* respectively in the same ratio, suppose *H* to be such a point on *AO* that *AH : HO=ag :go;*

and inscribe in the segment *BA B'* "in the recognised manner" a figure such that, if *F* he its centre of gravity,

*GF<GH.* [Prop. 6,  
Inscribe in the segment *bab'* "in the recognised manner" a similar figure; then, if *f* be the centre of gravity of this figure,

*ag <af.* [Prop. 5]  
And, by Prop. 3, *af :fo= AF : FO.*

But *AF : FO <AH : HO*

*<ag : go,* by hypothesis.

Therefore *af : fo <ag : go;* which is impossible.

It follows that *G, g* cannot but divide *AO,* ao in the same ratio.

**PROPOSITION** 8

*If AO be the diameter of a parabolic segment, and G its centre of gravity, then*

*AG=-3-GO.*

Let the segment be *BAB'.* Inscribe the triangle *ABB'* "in the recognised manner," and let *F* be its centre of gravity.

Bisect *AB, AB'* in *D, D',* and draw *DQ, D'Q'* parallel to *OA* to meet the curve, so that *QD, Q'D'* are the diameters of the segments *AQB, AQ'B'* respectively.

Let *II, H'* be the centres of gravity of the segments *AQB, AQ'B'* respective­ly. Join *QQ', HH'* meeting *AO* in V, *K* respectively.

*K* is then the centre of gravity of the two segments *AQB, AQ'B'* taken together.

Now AG : GO =QH : *HD,* [Prop. 7]

whence *AO : OG =QD : HD.*

But *AO =4QD [as* is easily proved by means of Lemma (3), p. 511].

Therefore *OG=4.11D,*

and, by subtraction, *AG=4QH.*

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Also, by Lemma (2), *QQ'* is parallel to *BB'* and therefore to *DD'.* It follows

from Prop. 7 that *HH'* is also parallel to *QQ'* or *DD',*

and hence *QH=VK.*

Therefore *AG = 4V K,*

and *AV+KG=3VK.*

Measuring *V L* along *V K* so that VL = V, we have

*KG =3LK. (1)*

Again *AO =4AV* [Lemma (3)]  
=3AL, since *AV =* 31/1,

whence *AL = 3AO =OF.* (2)

Now, by I. 6, 7,

P *ABB' :* (sum of segmts. *AQB, AQ'B') = KG : GP,*

and *ABB' =3(sum* of segments A*QB, AQ'B')* [since the segment *ABB'* is equal to *3-.ABB' (Quadrature of the Parabola,* Props. 17, 241].

Hence *KG =* 30F.

But *KG =3LK,* from (1) above.

Therefore *LF=LK+KG-FGF*

*=5GF.*

And, from (2),

*LF = (AO — A L OF) =* 1.40=0F.

Therefore *OF = 5GF,*

and *OG =6GF.*

But *AO* = *3OF = 15GF*Therefore, by subtraction,

*AG =9GF*

*=IGO.*

PROPOSITION 9 (LlimmA)

*If a, b, c, d be four lines in continued proportion and in descending order of magni-*

*tude, and if*

*d : (a — d) =x :(a— c),*

and (2a+4b-}-6c-E3d) : (5a+ 10b+ 10c-1-5d) *=y : (a— c),*

*it is required to prove that*

x+y=\*a.

[The following is the proof given by Archimedes, with the **A**

only difference that it is set out in algebraical instead of geo-

metrical notation. This is done in the particular case simply

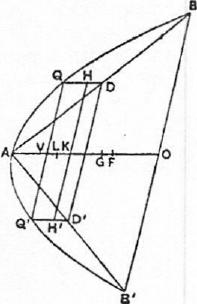
in order to make the proof easier to follow. Archimedes exhi- **-r**

bits his lines in the figure reproduced in the margin, but, now 0 that it is possible to use algebraical notation, there is no advan­tage in using the figure and the more cumbrous notation which

only obscures the course of the proof. The relation between -o

Archimedes' figure and the letters used below is as follows:

AB=a, 113=--b, AB=c, EB=d, ZII=x, He=y, 6,0=z.]



We have a\_ b\_c (1)

**B**

ON TILE EQUILIBRIUM OF PLANES II 115

whence

and therefore

Now

a—b *b—c c—d*

*=*

*d '*

*a—b b—c =a =b =c*

* *-———*

*b—c— c—d b c d*

2(a+b) *a+b* a+b *b a—c b—c a—c*

*=. —*

2c *c b c b—c c—d= c—d*And, in like manner,

*b+c \_b+c c a—c*

*c c—d*

It follows from the last two relations that

*a—c \_2a-F3b+c*

*c—d 2c+d*

Suppose *z* to be so taken that

2a+4b-}-4c-i-2d \_a—c

*2c+d*

so that *z<(c—d).*

Therefore *a—c+z* 2a-1-4b-1-6c-1-3d

a *—c* - 2(a+d)--1-4(b+c)•

And, by hypothesis,

*a—c* 5(a+d)-1-10(b+c)

y 2a-F4b1-6c+3d '

so that

a *—c+z 5(a+d)+10(b+c)* 5

y *2(a+d) -1-4(b+c) =* 2

Again, dividing (3) by (4) crosswise, we obtain

2a+3b-l-c

*c—d 2(a+d)+4(b+c)'*

whence *c—d—z \_*  b-4-3c-1-2d

*c—d 2(a+d)+4(b+c)*

But, by (2),

*c—d a —b* 3(b *—c) 2(c—d)*

*= =*

*d b 3c 2d '*

so that c—d\_(a—b)-1-3(b—c)+2(c—d)

*d b+3c-1-2d*

Combining (6) and (7), we have

*c—d—z \_(a—b)-1-3(b—c)+2(c—d)*

*d 2(a+d) +4(b+c)*

whence

*c—z*  3a+6b+3c

*d —- 2(a+d)+4(b+r)*

And, since [by (I)]

|  |  |  |
| --- | --- | --- |
| we have whence | *c—d \_b—c \_a—b  c+d— b+c— a+b'*  *c—d c+d*  *=*  *a—c b+c+a-Fb'*  *a—d a+2b-1-2c+d 2(a+d)-1-4(b+c) a—c a+2b+c =* 2(a+c)-1-4b | (9) |

-3-(a—c) 412(a-1-0+4bl '

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Thus a—d 2(a+d)-1-4(b+c)

and therefore, by hypothesis,

*d* 2(a+d)--1-4(b+c)

*x* 4{2(a+c)+41)}

c—z 3a-1-6b1-3c

But, by (8),

*d 2(a+d)+4(b+c)'*

and it follows, *ex aequali,* that

c—z 3(a+c)+6b 5 3 5

And, by (5), Therefore

*x* \*{2(a+c)+4b} 3 22'  
a—c+z 5

y 2.  
5 a   
2 x-Ey'

or x+y=*\*a.*

PROPOSITION 10

*If PP'B'B be the portion of a parabola intercepted between two parallel chords PP', BB' bisected respectively in N, 0 by the diameter ANO (N being nearer than 0 to A, the vertex of the segments), and if NO be divided into five equal parts of which LM is the middle one (L being nearer than. M to N), then, if G be a point on LM such that*

*LG :GM =BO' •(2PN+BO) : PN2 •(2BO-FPN),*

G *will be the centre of gravity of the area PP'B'B.*

Take a line *ao* equal to *A 0,* and an on it equal to *AN.* Let *p, q* be points on the line *ao* such that

*ao : aq=aq : an, (1)*

*ao : an=aq : ap,* (2)

[whence *ao : aq = aq : an = an : ap,* or *ao, aq, an, ap* are lines in continued pro-

portion and in descending order of magnitude].

Measure along *GA* a length *GF* such that

*op : ap =OL :GF.* (3)

Then, since *PN, BO* are ordinates to *ANO,*

*B02 : PN2 =AO : AN*

*=ao : an*

*= ao2 : aq2,* by (1),

so that *BO : PN = ao : aq,* (4)

and *B03 : PN3=ao3 :*

*= (ao : aq) -(aq : an) .(an : ap)*

*=ao : ap.* (5)

Thus (segment *BAB') :* (segment *PAP')*

*= PBAB' : LPAP'*

*= B03 : PN3*

*=ao : ap,*

whence

(area *PP'B'B) :* (segment *PAP') =op : ap*

ON THE EQUILIBRIUM OF PLANES **II**

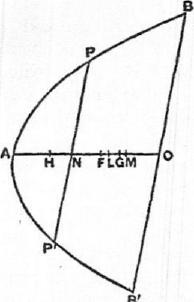
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*=OL : OF,* by (3),

=t,,ON : *OF.* (6)

Now

1302 .(2PN+ *BO) : B03*



*= (2PN +BO) : BO*

*=(2aq+ao) : ao,* by (4),

*1303 : PN3*

*=ao : ap,* by (5),

and

**a n a** *PN3 : PN2 •(2130+PN)*

*=PN : (2B0+PN)*

*=aq : (2ao+aq),* by (4),

*=ap :* (2an+ap), by (2).

Hence, *ex aequali,*

*B02 .(2PN+BO) : PN2 •*

*(280+PN) (2aq+ao) :*

(2an+ap),

so that, by hypothesis,

*LG : GM = (2aq+ ao) :* (2an+ap).

*Componendo,* and multiplying the antecedents by 5,

*ON : GM =* 15(ao±ap)+10(aq+an)) : (2an+ap).

But

*ON : OM =5* : 2= 5(ao-Fap)+10(aq+an)) : f2(ao+ap)-1-4(aq-l-an)).

It follows that

*ON :00=* { 5(ao+ap)±10(aq-Fan) : (2ao-1-4aq+6an-F3ap).

Therefore

(2ao 6an+ 3ap) : 5(ao ap) 10(aq+ an) } = *OG : ON*

*=00 :on.*

And ap : *(ao—ap) = ap :.op*

*= GF OL,* by hypothesis,

*=GF :ion,*

while *ao, aq, an,* ap are in continued proportion.

Therefore, by Prop. 9,

*GF+OG=OF =\*ao = t-OA.*

Thus *F* is the centre of gravity of the segment *BA B'.* [Prop. 8]  
Let *H* be the centre of gravity of the segment *PAP',* so that *AH =\*AN.*

And, since AF =\*A0,

we have, by subtraction, *HF = SON.*

But, by (6) above,

(area *PP'11'.8) :* (segment *PAP') = \*ON : GF*

*=HF : FO.*

Thus, since *F, H* are the centres of gravity of the segments *BAB', PAP'* re-

spectively, it follows [by I. 6, 7] that *G* is the centre of gravity of the area

*PP'B'B.*

**THE SAND-RECKONER**

**"THERE** are some, King Gelon, who think that the number of the sand is infinite in multitude; and I mean by the sand not only that which exists about Syracuse and the rest of Sicily but also that which is found in every region whether inhabited or uninhabited. Again there are some who, without regard­ing it as infinite, yet think that no number has been named which is great enough to exceed its multitude. And it is clear that they who hold this view, if they imagined a mass made up of sand in other respects as large as the mass of the earth, including in it all the seas and the hollows of the earth filled up to a height equal to that of the highest of the mountains, would be many times further still from recognising that any number could be expressed which ex­ceeded the multitude of the sand so taken. But I will try to show you by means of geometrical proofs, which you will be able to follow, that, of the numbers named by me and given in the work which I sent to Zeuxippus, some exceed not only the number of the mass of sand equal in magnitude to the earth filled up in the way described, but also that of a mass equal in magnitude to the universe. Now you are aware that `universe' is the name given by most astron­omers to the sphere whose centre is the centre of the earth and whose radius is equal to the straight line between the centre of the sun and the centre of the earth. This is the common account (Ta.ypackIleva), as you have heard from astronomers. But Aristarchus of Samos brought out a book consisting of some hypotheses, in which the premisses lead to the result that the universe is many times greater than that now so called. His hypotheses are that the fixed stars and the sun remain unmoved, that the earth revolves about the sun in the circumference of a circle, the sun lying in the middle of the orbit, and that the sphere of the fixed stars, situated about the same centre as the sun, is so great that the circle in which he supposes the earth to revolve bears such a propor­tion to the distance of the fixed stars as the centre of the sphere bears to its surface. Now it is easy to see that this is impossible; for, since the centre of the sphere has no magnitude, we cannot conceive it to bear any ratio whatever to the surface of the sphere. We must however take Aristarchus to mean this: since we conceive the earth to be, as it were, the centre of the universe, the ratio which the earth bears to what we describe as the `universe' is the same as the ratio which the sphere containing the circle in which he supposes the earth to revolve bears to the sphere of the fixed stars. For he adapts the proofs of his results to a hypothesis of this kind, and in particular he appears to suppose the magnitude of the sphere in which he represents the earth as moving to be equal to what we call the `universe.'

"I say then that, even if a sphere were made up of the sand, as great as Aristarchus supposes the sphere of the fixed stars to be, I shall still prove that,

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THE SAND-RECKONER 119

of the numbers named in the *Principles,'* some exceed in multitude the number of the sand which is equal in magnitude to the sphere referred to, provided that the following assumptions be made."

1. *"The perimeter of the earth is about* 3,000,000 *stadia and not greater.*

"It is true that some have tried, as you are of course aware, to prove that the said perimeter is about 300,000 stadia. But I go further and, putting the mag­nitude of the earth at ten times the size that my predecessors thought it, I suppose its perimeter to be about 3,000,000 stadia and not greater."

1. *"The diameter of the earth is greater than the diameter of the moon, and the*

*diameter of the sun is greater than the diameter of the earth.*

"In this assumption I follow most of the earlier astronomers."

1. *"The diameter of the sun is about* 30 *times the diameter of the moon and not greater.*

"It is true that, of the earlier astronomers, Eudoxus declared it to be about nine times as great, and Pheidias my father twelve times, while Aristarchus tried to prove that the diameter of the sun is greater than 18 times but less than 20 times the diameter of the moon. But I go even further than Aristar-chus, in order that the truth of my proposition may be established beyond dispute, and I suppose the diameter of the sun to be about 30 times that of the moon and not greater."

1. *"The diameter of the sun is greater than the side of the chiliagon inscribed in*

*the greatest circle in the (sphere of the) universe.*

"I make this assumption because Aristarchus discovered that the sun ap-

peared to be about **7** s 0th part of the circle of the zodiac, and I myself tried, by  
a method which I will now describe, to find experimentally (A *czyuciiis)* the angle subtended by the sun and having its vertex at the eye."

[Up to this point the treatise has been literally translated because of the historical interest attaching to the *ipsissima verba* of Archimedes on such a subject. The rest of the work can now be more freely reproduced, and, before proceeding to the mathematical contents of it, it is only necessary to remark that Archimedes next describes how he arrived at a higher and a lower limit for the angle subtended by the sun. This he did by taking a long rod or ruler, fastening on the end of it a small cylinder or disc, pointing the rod in the direc­tion of the sun just after its rising (so that it was possible to look directly at it), then putting the cylinder at such a distance that it just concealed, and just failed to conceal, the sun, and lastly measuring the angles subtended by the cylinder. He explains also the correction which he thought it neces­sary to make because "the eye does not see from one point but from a certain area."'

The result of the experiment was to show that the angle subtended by the diameter of the sun was less than TiTth part, and greater than 3.6th part, of a right angle.

*To prove that (on this assumption) the diameter of the sun is greater than the side of a chiliagon, or figure with* 1000 *equal sides, inscribed in a great circle of the "universe."*

Suppose the plane of the paper to be the plane passing through the centre of the sun, the centre of the earth and the eye, at the time when the sun has

IA lost work of Archimedes.

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just risen above the horizon. Let the plane cut the earth in the circle *EHL* and the sun in the circle *FKG,* the centres of the earth and sun being *C,* 0 respec­tively, and *E* being the position of the eye.

Further, let the plane cut the sphere of the "universe" (i.e. the sphere whose centre is *C* and radius *CO)* in the great circle *AOB.*

Draw from *E* two tangents to the circle *FKG* touching it at *P,* Q, and from *C* draw two other tangents to the same circle touching it in *F, G* respectively.

Let CO meet the sections of the earth and sun in *H, K* respectively; and let *CF, CG* produced meet the great circle *AOB* in *A, B.*

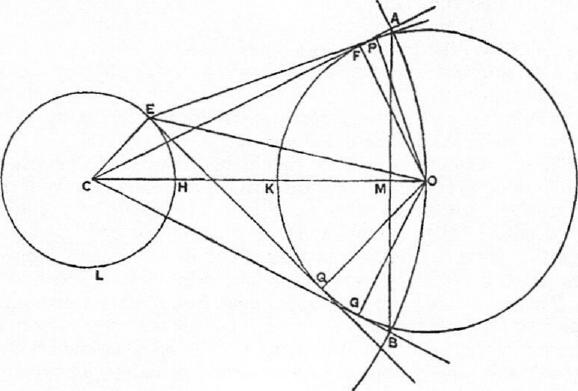
Join *EO, OF, OG, OP, OQ, AB,* and let *AB* meet *CO* in *M.*

Now *CO > EO,* since the sun is just above the horizon.

Therefore L *PEQ> L FCG.*

And *ZPEQ>vh.R1*

but <thn f where *R* represents a right angle.



Thus L *FCC <-14-4-R,* a *fortiori,*

and the chord *AB* subtends an arc of the great circle which is less than 33-hrth

of the circumference of that circle, i.e.

*AB <* (side of 656-sided polygon inscribed in the circle).

Now the perimeter of any polygon inscribed in the great circle is less than

5.74-CO. [Cf. *Measurement of a circle,* Prop. 3.]

Therefore *AB : CO* <11 : 1148,

and, *a fortiori, AB <TL6C0.* (a)  
Again, since CA *= CO,* and *AM* is perpendicular to *CO,* while *OF* is perpen-

dicular to *CA, AM = OF.*

Therefore *AB = 2AM =* (diameter of sun).

Thus (diameter of sun) <rhCO, by (a),

and, a *fortiori,* (diameter of earth) <11000. [Assumption 2]

Hence *CH +0K <T6CO,*

so that *IIK>MCO,*

THE SAND-RECKONER **121**

or *CO* ***HK <100 :*** 99.

And *CO>CF,*

while ***HK <EQ.***

Therefore ***CF : EQ*** <100 : 99. ($)  
Now in the right-angled triangles ***CFO, EQO,*** of the sides about the right

angles, ***OF =OQ,*** but ***EQ <CF*** (since ***EO <CO).***

Therefore L ***OEQ : L OCF> CO : ED,***

but *<CF* ***: EQ.'***

Doubling the angles,

*L****PEQ : LACB<CF : EQ***

***<100*** : 99, by (0) above.

But L ***PEQ >*** thR, by hypothesis.

Therefore L ***ACB>y-ogt6R***

***> YhR•***

It follows that the arc ***AB*** is greater than Agth of the circumference of the

great circle ***AOB.***

Hence, *a fortiori,*

*AB>* (side of chiliagon inscribed in great circle),

and *AB* is equal to the diameter of the sun, as proved above.

*The following results can now be proved:*

*(diameter of "universe") <10,000 (diameter of earth),*

*and (diameter of "universe") <10,000,000,000 stadia.*

1. Suppose, for brevity, that *du* represents the diameter of the "universe," *d,* that of the sun, de that of the earth, and *d,* that of the moon.

By hypothesis, ***da>*** 30d,, [Assumption 3]

and *de>d,u;* [Assumption 2]

therefore ***di<*** 30de.

Now, by the last proposition,

***d,>*** (side of chiliagon inscribed in great circle),

so that (perimeter of chiliagon) <1000d,

<30,000de.

But the perimeter of any regular polygon with more sides than 6 inscribed

in a circle is greater than that of the inscribed regular hexagon, and therefore

greater than three times the diameter. Hence

(perimeter of chiliagon) > 3d..

It follows that *du<10,000de.*

1. (Perimeter of earth) >3,000,000 stadia. [Assumption 1]

and (perimeter of earth) >34.

Therefore *d,<1,000,000* stadia,

whence *du<10,000,000,000* stadia.

*Assumption* 5

Suppose a quantity of sand taken not greater than a poppy-seed, and sup­pose that it contains not more than 10,000 grains.

'The proposition here assumed is of course equivalent to the trigonometrical formula which states that, if a, fi are the circular measures of two angles, each less than a right angle, of which a is the greater, then

tan ***a> a>*** sin ***a***

tan *t3* sin )9

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Next suppose the diameter of the poppy-seed to be not less than isth of a finger-breadth.

**ORDERS AND PERIODS OF NUMBERS**

1. We have traditional names for numbers up to a myriad (10,000); we can therefore express numbers up to a myriad myriads (100,000,000). Let these numbers be called numbers of the *first order.*

Suppose the 100,000,000 to be the unit of the *second order,* and let the *second order* consist of the numbers from that unit up to (100,000,000)2.

Let this again be the unit of the *third order* of numbers ending with (100,­000,000)'; and so on, until we reach the 100,000,000th *order* of numbers ending with (100,000,000)100.000.00°, which we will call ***P.***

1. Suppose the numbers from 1 to ***P*** just described to form the *first period.* Let *P* be the unit of the *first order of the second period,* and let this consist of the numbers from ***P*** up to 100,000,000P.

Let the last number be the unit of the *second order of the second period,* **and** let this end with (100,000,000)2P.

We can go on in this way till we reach the 100,000,000th *order of the second period* ending with (100,000,000)100.000.000 P, or P2.

1. Taking P2 as the unit of the *first order of the third period,* we proceed in the same way till we reach the 100,000,000th *order of the third period* ending with *P3.*
2. Taking P3 as the unit of the *first order of the fourth period,* we continue the same process until we arrive at the 100,000,000th *order of the* 100,000,000th *period* ending with P10°•0°°•0°°. This last number is expressed by Archimedes as "a myriad-myriad units of the myriad-myriad-th order of the myriad-myriad-th period (al µuptarctoµuptooras *irept6r5ou AvinatacrtivpLoarCo IcptOph.7)v* µvptat AuptAbes)," which is easily seen to be 100,000,000 times the product of (100,000,000)

99.00,9Q0 and p99,999,999, i.e. ploopo,000.  
**OCI'ADS**

Consider the series of terms in continued proportion of which the first **is** 1 and the second 10 [i.e. the geometrical progression 1, 10', 102, 103, • • •]. The *first octad* of these terms *[i.e. 1,* 10', 102, • • 401 fall accordingly under the *first order of the first period* above described, the *second octad* [i.e. 108, 109, • • •10'5] under the *second order of the first period,* the first term of the octad being the unit of the corresponding order in each case. Similarly for the *third octad,* and so on. We can, in the same way, place any number of octads.

**THEOREM**

*If there be any number of terms of a series in continued proportion, say A1, A2,* **A** 3, • • • *Am, • • • An, • • • Am+n\_i, • • • of which A1=1, A2=10* [so that the series forms the geometrical progression 1, 10', 102, • • •10m--1, • • -10-1,

- -], and *if any two terms as Am,* ***An*** *be taken and multiplied, the product Am •A n will be a term in the same series and will be as many terms distant from An as Am is distant from Al; also it will be distant from Al by a number of terms less by one than the sum of the numbers of terms by which Am and* ***An*** *respectively are distant from* A1.

Take the term which is distant from ***An*** by the same number of **terms as *A.***

THE SAND-RECKONER 123

is distant from *Al.* This number of terms is m (the first and last being both counted). Thus the term to be taken is m terms distant from *A.,* and is there­fore the term

We have therefore to prove that

* *An= Am÷n-i.*

Now terms equally distant from other terms in the continued proportion are

Proportional.

Thus Am Am+n-i

**Al An**

But *A„,= Am Ai,* since *A1=1.*

Therefore Arn+n-l= •An. (1)  
The second result is now obvious, since *Am* is m terms distant from A1, An is n terms distant from A1, and A„+,.1 is (m+n — 1) terms distant from **Al.**

**APPLICATION TO THE NUMBER OF THE SAND**

By Assumption 5 [p. 523],

(diam. of poppy-seed) <-1-1-0-(finger-breadth);

and, since spheres are to one another in the triplicate ratio of their diameters,

it follows that

(sphere of diam. 1 finger-breadth)>64,000 poppy-seeds

> 64,000 X 10,000

>640,000,000

>6 units of *second* grains

*order+* 40,000,000 of

units of *first order* sand.

(a *fortiori) <* 10 units of *second*

*order* of numbers.

We now gradually increase the diameter of the supposed sphere, multiplying

it by 100 each time. Thus, remembering that the sphere is thereby multiplied

by 1003 or 1,000,000, the number of grains of sand which would be contained

in a sphere with each successive diameter may be arrived at as follows.

*Diameter of sphere. Corresponding number of grains of sand.*

1. 100 finger-breadths <1,000,000 X 10 units of *second order* < (7th term of series) X (10th term of series)

<16th term of series [i.e. 1015]  
<I107 or] 10,000,000 units of the *second order.*

1. 10,000 finger-breadths <1,000,000 X (last number) < (7th term of series) X (16th term)

<22nd term of series [i.e. 10i1]  
<[105 or] 100,000 units of *third order.*

1. 1 stadium <100,000 units of *third order. (<10,000* finger-breadths)
2. 100 stadia <1,000,000 X (last number) < (7th term of series) X(22nd term)

<28th term of series [1029  
<[10g or] 1,000 units of *fourth order.*

1. 10,000 stadia <1,000,000 X (last number) <(7th term of series) X(28th term)

<34th term of series [1039  
<10 units of *fifth order.*

|  |  |  |  |
| --- | --- | --- | --- |
| 124 | imam() stadia | ARCHIMEDES  <(7th term of series)X(34th term) |  |
|  |  | <40th term | [1039] |
|  |  | <[107 or] 10,000,000 units of *fifth order.* |  |
|  | 100,000,000 stadia | < (7th term of series) X (40th term) |  |
|  |  | <46th term | (1a6] |
|  |  | <[1O or] 100,000 units of *sixth order.* |  |
|  | 10,000,000,000 stadia | <(7th term of series) X (46th term) |  |
|  |  | <52nd term of series | [10,1] |
|  |  | <[103 or] 1,000 units of *seventh order.* |  |

But, by the proposition above [p. 523],

(diameter of "universe") <10,000,000,000 stadia.

Hence *the number of grains of sand which could be contained in a sphere of the*

*size of our "universe" is less than 1,000 units of the seventh order of numbers*

[or 10].

From this we can prove further that a *sphere of the size attributed by Aristar-*

*chus to the sphere of the fixed stars would contain a number of grains of sand less*

*than* 10,000,000 *units of the eighth order of numbers* [or 10'6'7=1063].

For, by hypothesis,

(earth) : ("universe") = ("universe") : (sphere of fixed stars).

And [p. 523]

(diameter of "universe") <10,000 (diam. of earth);

whence

(diam. of sphere of fixed stars) <10,000 (diam. of "universe").

Therefore

(sphere of fixed stars) < (10,000)3 •("universe").

It follows that the number of grains of sand which would be contained in a

sphere equal to the sphere of the fixed stars

< (10,000)3X 1,000 units of *seventh order*

< (13th term of series) X (52nd term of series)

<64th term of series [i.e. 1063]

<[107 or] 10,000,000 units of *eighth order* of numbers.

CONCLUSION.

"I conceive that these things, King Gelon, will appear incredible to the great majority of people who have not studied mathematics, but that to those who are conversant therewith and have given thought to the question of the distances and sizes of the earth, the sun and moon and the whole universe, the proof will carry conviction. And it was for this reason that I thought the sub­ject would he not inappropriate for your consideration."

QUADRATURE OF THE PARABOLA

**"ARCHIMEDES to DOSITEIEIIS** greeting.

"When I heard that Conon, who was my friend in his lifetime, was dead, but that you were acquainted with Conon and withal versed in geometry, while I grieved for the loss not only of a friend but of an admirable mathematician, I set myself the task of communicating to you, as I had intended to send to Conon, a certain geometrical theorem which had not been investigated before but has now been investigated by me, and which I first discovered by means of mechanics and then exhibited by means of geometry. Now some of the earlier geometers tried to prove it possible to find a rectilineal area equal to a given circle and a given segment of a circle; and after that they endeavoured to square the area bounded by the section of the whole cone and a straight line, assuming lemmas not easily conceded, so that it was recognised by most people that the problem was not solved. But I am not aware that any one of my predecessors has attempted to square the segment bounded by a straight line and a section of a right-angled cone [a parabola], of which problem I have now discovered the solution. For it is here shown that every segment bounded by a straight line and a section of a right-angled cone [a parabola] is four-thirds of the triangle which has the same base and equal height with the segment, and for the demonstration of this property the following lemma is assumed: that the excess by which the greater of (two) unequal areas exceeds the less can, by being added to itself, be made to exceed any given finite area. The earlier geometers have also used this lemma; for it is by the use of this same lemma that they have shown that circles are to one another in the duplicate ratio of their diameters, and that spheres are to one another in the triplicate ratio of their diameters, and further that every pyramid is one third part of the prism which has the same base with the pyramid and equal height; also, that every cone is one third part of the cylinder having the same base as the cone and equal height they proved by assuming a certain lemma similar to that afore­said. And, in the result, each of the aforesaid theorems has been accepted no less than those proved without the lemma. As therefore my work now pub­lished has satisfied the same test as the propositions referred to, I have written out the proof and send it to you, first as investigated by means of mechanics, and afterwards too as demonstrated by geometry. Prefixed are, also, the ele­mentary propositions in conics which are of service in the proof. Farewell."

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PROPOSITI ON 1

*If from a point on a parabola a straight line be drawn which is either itself the axis*

*or parallel to the axis, as PV, and if QQ' be a chord parallel to the tangent to the*

*parabola at P and meeting PV in V, then*

*QV = VQ'.*

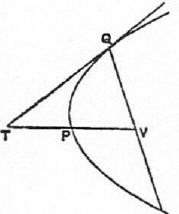
Conversely, *if QV =17(2', the chord* QQ' *will be parallel to the tangent at P.*



PROPOSITION 2

*If in a parabola QQ' be a chord parallel to the tangent at P, and if a straight line be drawn through P which is either itself the axis or parallel to the axis, and which meets* QQ' *in V and the tangent at Q to the parabola in T, then*

*PV=PT.*



PROPOSITION 3

*If from a point on a parabola a straight line be drawn which is either itself the axis or parallel to the axis, as PV, and if from two other points Q, Q' on the parabola straight lines be drawn parallel to the tangent at P and meeting PV in V, V'*

*respectively, then PV : PV' =QV' :Q'V".*

*"And these propositions are proved in the elements of conics."'*

PROPOSITION 4

*If Qq be the base of any segment of a parabola, and P the vertex of the segment,* and

*if the diameter through any other point R meet Qq in 0 and QP (produced if*

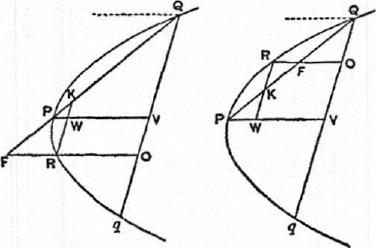
*necessary) in F, then QV : VO = OF :FR.*Draw the ordinate *RW* to *PV,* meeting *QP* in *K.*

Then *PV : PW = QV' :* RIV2;

whence, by parallels, *PQ : PK =PQ2 : PF2.*'i.e. in the treatises on conics by Euclid and Aristaeus.

QUADRATURE OF THE PARABOLA 127

In other words, *PQ, PF, PK* are in continued proportion; therefore



*PQ :PF =PF :PK*

*=PQ+PF :PF-I-PK*

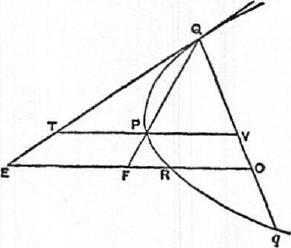
*=QF : KF.*

Hence, by parallels, *QV : VO =OF : FR.*

**PROPOSITION** 5

*If Qq be the base of any segment of a parabola, P the vertex of the segment, and PV its diameter, and if the diameter of the parabola through any other point R meet Qq in 0 and the tangent at Q in E, then*

*QO :0q=ER : RO.*



Let the diameter through *R* meet *QP*

in *F.*

Then, by Prop. 4,

*QV : VO =OF :FR.*

Since *QV =V q,* it follows that

*QV : q0 =OF : OR. (1)*

Also, *if VP* meet the tangent in 7',

*PT =PV,* and therefore *EF =OF.*

Accordingly, doubling the antecedents

in (1), we have

*Qq : q0 =OE : OR,*

whence QO : *0q=ER : RO.*

**PROPOSITIONS** 6, *7'*

*Suppose a lever AOB placed horizontally and supported at its middle point 0. Let* a *triangle BCD in which the angle C is right or obtuse be suspended from B and 0, so that C is attached to 0 and CD is in the same vertical line with 0. Then, if P be such an area as, when suspended from A, will keep the system in equilibrium,*

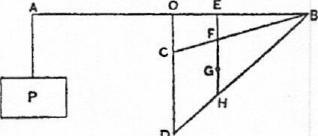
*P=APBCD.*

'In Prop. 6 Archimedes takes the separate case in which the angle *BCD* of the triangle is a right angle so that *C* coincides with *0* in the figure and *F* with *E.* He then proves, in Prop. 7, the same property for the triangle in which *BCD* is an obtuse angle, by treating the tri­angle as the difference between two right-angled triangles *BOD, BOG* and using the result of Prop. 6. I have combined the two propositions in one proof, for the sake of brevity. The same remark applies to the propositions following Props. 6, 7.

**128** ARCIIINIEDES

Take a point *E* on 0.8 such that *BE =20E,* and draw *ERR* parallel to *OCD* meeting *BC, BD* in *F, H* respectively.

Let *G* be the middle point of *FH.*



Then *G* is the centre of gravity of the triangle *BCD.*

Hence, if the angular points *B, C* be set free and the triangle be suspended by attaching *F* to *E,* the triangle will hang in the same position as before, because *EFG* is a vertical straight line. "For this is proved."'

Therefore, as before, there will be equilibrium.

Thus *P : LBCD=OE : AO*

*=1 :*3,

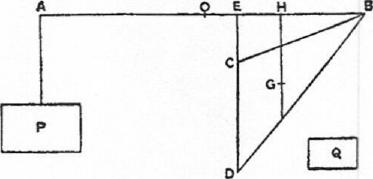
or *P= ,/BCD.*

**PROPOSITIONS** 8, 9

*Suppose a lever AOB placed horizontally and supported at its middle point 0. Let a triangle BCD, right-angled or obtuse-angled at C, be suspended from the points B, B on OB, the angular point C being so attached to E that the side CD is in the same vertical line with E. Let* Q *be an area such that*

*AO :OE= LBCD : Q.*

*Then, if an area P suspended from A keep the system in equilibrium, P < PBCD but>Q.*



Take *G* the centre of gravity of the triangle *BCD,* and draw *GH* parallel to *DC,* i.e. vertically, meeting *BO* in *H.*

*We* may now suppose the tri­angle *BCD* suspended from *H,* and, since there is equilibrium,

*LBCD* : *P= AO : OH,* (1)

whence *P<ABCD.*

Also *LBCD :Q= AO : OE.*

Therefore, by (1), *LBCD :Q> LBCD :P,*

and *P > Q.*

*PROPosiTIoNS 10, 11*

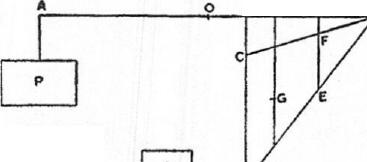
*Suppose* a *lever AOB placed horizontally and supported at 0, its middle point. Let CDEF be a trapezium which can be so placed that its parallel sides CD, FE are vertical, while C is vertically below 0, and the other sides CF, DE meet in B. Let EF meet BO in H, and let the trapezium be suspended by attaching F to H and C to 0. Further, suppose Q to be an area such that*

*AO : OH = (trapezium CDEF) : Q.*

*Then, if P be the area which, when suspended from A, keeps the system. in equi­librium,*

*P<Q.*

'Doubtless in the lost book *rept* ruyi,-,v.



QUADRATURE OF THE PARABOLA 129

*The same is true in the particular case where the angles at C, F are right, and*

*consequently C, F coincide with 0, H respectively.*

Divide *0II* in *K* so that

*(2CD+FE) : (2FE+CD)=11K : KO.*

Draw *KG* parallel to *OD,* and

**a** let *G* be the middle point of the

portion of *KG* intercepted with­in the trapezium. Then *G* is the centre of gravity of the trape­zium [On *the equilibrium of planes,* I. 15].

Thus we may suppose the tra­pezium suspended from *K,* and

the equilibrium will remain undisturbed.

Therefore *AO : OK =* (trapezium *CDEF) : P,* and, by hypothesis, *AO : OH =* (trapezium *CDEF) : Q.* Since *OK <011,* it follows that

*P <Q.*

*PRorosmoNs* 12, 13

*If the trapezium CDEF be placed as in the last propositions, except that CD is*

*vertically below a point L on OB instead of being below 0, and the trapezium is*

*suspended from L, H, suppose that Q, R are areas such that*

*AO : 011 = (trapezium CDEF) :Q,*

*AO : OL= (trapezium CDEF) : R.*

*If then an area P suspended*

*from A keep the system in equilib-*

**L K**

*rium,*

*P>R but<Q.*

Take the centre of gravity of the trapezium, as in the last propositions, and let the line through G parallel to *DC* meet OB in *K.*

Then we may suppose the tra­pezium suspended from *K,* and there will still be equilibrium.

Therefore (trapezium *CDEF) : P =AO :OK.*

Hence (trapezium *CDEF) : P>* (trapezium *CDEF') : Q,*

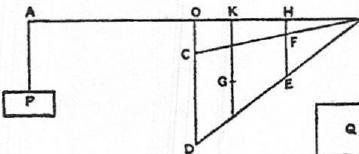
but < (trapezium *CDEF) : R.*

It follows that *P <Q* but *>R.*

**PROPOSITIONS** 14, 15

Let *Qq* be the base of any segment of a parabola. Then, if two lines be drawn from *Q, q,* each parallel to the axis of the parabola and on the same side of *Qq* as the segment is, either (1) the angles so formed at *Q, q* are both right angles, or (2) one is acute and the other obtuse. In the latter case let the angle at *q* be the obtuse angle.

Divide *Qq* into any number of equal parts at the points 01, 02, • • Draw



**a**

and

Q

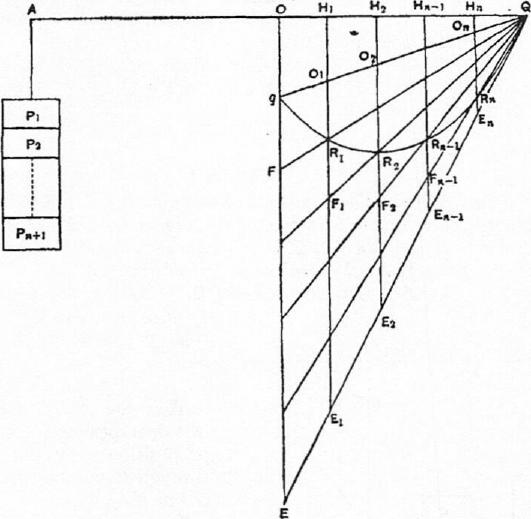
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through *q, 01,* 02, • • .0„ diameters of the parabola meeting the tangent at Q in *E, E1,* **E2, •** *•E„* and the parabola itself in *q; Ri,* **R2, • •** *•R„.* Join QR1, *QR2, QR„* meeting *qE, 01E1, 02E2, • • .0,,\_1E„\_1* in *F, F1, F2, • •F„\_1.*

Let the diameters *Eq, E101, • • • E „0„* meet, a straight line *QOA* drawn through Q perpendicular to the diameters in the points *0, III, H2, • • •II„* respectively. (In the particular case where *Qq* is itself perpendicular to the diameters *q* will coincide with *0, 01* with HI, and so on.)

*It is required to prove that*

1. *PEqQ<3(sum of trapezia P101, F102, • • .F...40„ and E„OuQ),*
2. *AEqQ>3(sum of trapezia 11102,R203, • • •R„\_10„ and AR„0.(2).*



Suppose *AO* made equal to OQ, and conceive Q0\_4 as a lever placed hori­zontally and supported at.O. Suppose the triangle *EqQ* suspended from *OQ* in the position drawn, and suppose that the trapezium *E01* in the position drawn is balanced by an area *P1* suspended from A, the trapezium E102 in the position drawn is balanced by the area **P2** suspended from A, and so on, the triangle *E7,0„Q* being in like manner balanced by P.+1.

Then P1+P2+ • • • *1-Pn+ilvill* balance the whole triangle *EqQ* as drawn, and

therefore P1+P2+ • • •+1)„4.1= A AEyQ: [Props. 0, 7]

Again *AO :0111=Q0 :011i*

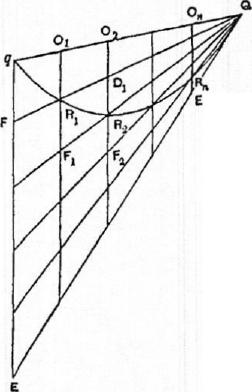
*=Qg gOi*

*=E101: oire,* [by means of Prop. 5)

= (trapezium E01) : (trapezium F01);

whence [Props. 10, 11] (F01) > P1.

QUADRATURE OF THE 1'ARAI3OI.A



131 (a)

(0) 13,

Next *AO :Nil= E101:01Ri*

*= (E102) : (1?102),*

while *AO :0H2=E202: 02R2*

*= (E102) : (F102);;*

and, since (a) and (g) are simultaneously true, we have, by Props. 12,

(F102) > ***P2>*** (R102).

Similarly it may be proved that

(F203) >P3> (R203),

and so on.

Lastly [Props. 8, 0] PE„0„Q>P7,1> ,L/?„0„Q. By addition, we obtain

1. (F01)-1-•(P102)± • • •+(F„.\_10.)-F PE„0„(2>P1-1-P2-1- • • •-i-Pn+i > A *PEW,*

or *,LEV2* <3(E01-1- F102-1- • • •+1•',•\_30„-h,LE,.O„Q*).*

1. (/602)-1- (/?203) + • • *cien\_.,o„)+ Rh0„Q* <P2+ ***P3+ • • • + Pn+1***

***<J****'****,+1'\_+ • • •*** *+P„ 0, a fortiori,*

*<1 LEW ,*

*or 6,14.1qQ>* 3(1602+11203+ • • • +/?„.\_10,,-1- *PR„0,(2).*

PitoPosirriox 16

*Suppose Qq to be the base of a parabolic segment, q being not more distant than* Q *from the vertex of the parabola. Draw through q the straight line qE parallel to the axis of the parabola to meet the tangent at Q in E. It is required to prore that (area of segment) = PEW.*

For, if not, the area of the segment must be either greater or less than 16,4Q.

T. Suppose the area of the segment greater than *AEgQ.* Then the excess can, if contin­ually added to itself, be made to exceed *AM.*

ri And it is possible to find a submultiple of the triangle *EqQ* less than the said excess of the segment over *IAEA*

Let the triangle *FqQ* be such a submultiple of the triangle *EqQ.* Divide *Eq* into equal parts each equal to *qP,* and let all the points of divi­sion including *F* be joined to Q meeting the parabola in RI, R2, • • *-R„* respectively. Through **RI, R2, • • •Rn draw di:Inlet CI'S** of the parabola meeting q(2 in 01, 02, • • *-0„* respectively.

Let. WI', meet Q112 in F.

Let 02/?2 meet (2111 in DI and QR3 in F2.

Let 03113 meet (1/?2 in 1)2 and *QM* in **F3,** and so on.

We have, by hypothesis,

*PV2 <* (area of segment) — A *PEW,*

or (area of segment)— L *Fq(2> ;PEW.* (a)  
Now, since all the parts of *qE,* as *qF* and the rest., are equal, *01R1=RiFi, 02D1=D1R2=R2F2,* and so on; therefore

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*AFqQ= (F01-F-R102+ D103+ • • .)*

*= (F01+ FiDi+F2D2+ • • • + F n\_ip„.\_1+* A EnR.(2)• ($)

But (area of segment) < (FOld-F102-1- • • • -1-1117,10.-1- *AE.0.(2)•*

Subtracting, we have

(area of segment) — A *Fq(2 < (R102+ R203+ • • • + R.-10.+* A *R.O.Q),*

whence, a *fortiori, by (a),*

*1AEqQ<(R102+R203+ • • • +Rn\_10.-f- AR.On(2)•*

But this is impossible, since [Props. 14, 15]

*AEqQ> (11102-1-R203+ • • •+ LR„0,,Q).*

Therefore (area of segment) > 1AE9Q.

II. If possible, suppose the area of the segment less than *1PEqQ.*

Take a submultiple of the triangle EqQ, as the triangle *FqQ,* less than the

excess of *1PEq(2* over the area of the segment, and make the same construc-

tion as before.

Since p *FqQ <I L —* (area of segment),

it follows that

*AFqQ-1-* (area of segment) <1 *PEqQ*

*<(F01+ F102+ • • • +F Z,E„0„Q).*

[Props. 14, 15]

Subtracting from each side the area of the segment, we have

*&'qQ <* (sum of spaces *qFRi, RiFiR2, • • • E.R.Q)*

*< (F01+ FiD14- • • •+Fn\_iD.-1+ AE.R.Q),* a *fortiori;*

which is impossible, because, by (0) above,

*6,17W = F03.+FiDi+ • • • + F .LE.R„(2.*

Hence (area of segment)<IPEqQ.

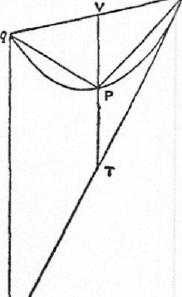
Since then the area of the segment is neither less nor greater than *IAEA*

it is equal to it.

PROPOSITION 17

It is now manifest that *the area of any segment of a parabola is four-thirds of the triangle which has the same base as the segment and equal height.*

Let *Qq* be the base of the segment, ***P*** its vertex. a  
Then *PQq* is the inscribed triangle with the same base as the segment and equal height.



Since ***P*** is the vertex of the segment, the diame­ter through *P* bisects *Qq.* Let *V* be the point of bi­section.

Let *VP,* and *0;* drawn parallel to it, meet the tan­gent at *Q* in 7', *E* respectively.

Then, by parallels,

*qE =2177',*

and *PV=PT,* [Prop. 2]

so that *VT* =2PV.

Hence *6.EqQ — 4 PPQq •*

But, by Prop. 16, the area of the segment is equal to

*EqQ.*

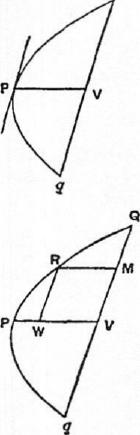
Therefore (area of segment)= *46,1)(2g.*

DEF. "In segments bounded by a straight line and any curve I call the

QUADRATURE OF TILE PARABOLA 133

straight line the *base,* and the *height* the greatest perpendicular drawn from the curve to the base of the segment, and the *vertex* the point from which the greatest perpendicular is drawn."

fl PROPOSITION 18



*If Qq be the base of a segment of a parabola, and V the middle point of Qq,* and *if the diameter through V meet the curve in P, then P is the vertex of the segment.*

For *Qq* is parallel to the tangent at *P* [Prop. 1]. Therefore, of all the perpendiculars which can be drawn from points on the segment to the base *Qq,* that from *P* is the greatest. Hence, by the definition, *P* is the vertex of the segment.

PROPOSITION 19

*If Qq be a chord of a parabola bisected in V by the diameter*

*PV, and if RM be a diameter bisecting QV in M, and R1V*

*he the ordinate from R to PV, then*

*PV = ARM.*

For, by the property of the parabola,

*PV :* PW=QV2 : *R1V2*

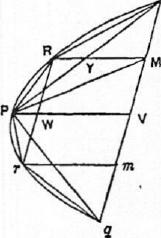
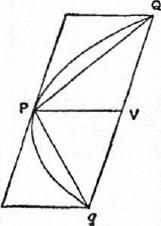
= 4RIV2 : RTV2,

so that *PV =* 4/31V,

whence *PV =1-R31.*

PROPOSITION 20

*If Qq be the base, and P the vertex, of* a *parabolic segment, then the triangle PQq is greater than half the segment PQq.*



For the chord Qq is parallel to the tangent at *P,* and the triangle *PQq* is half the parallelogram formed *by Qq,* the tangent at *P,* and the diameters through *Q, q.*

Therefore the triangle *PQq* is greater than half the segment.

COR. It follows that *it is possible to inscribe in the seg­ment a polygon such that the segments left over are together less than any assigned area..*

PROPOSITION 21

*If Qq be the base, and 1' the vertex, of any parabolic seg­ment, and if R be the vertex of the segment cut off by PQ, then*

*PPQg = 8PPRQ.*

The diameter through *R* will bisect the chord *PQ,* and therefore also *QV,* where *PV* is the diameter bisect­ing *Qq.* Let the diameter through *I?* bisect *PQ* in Y and *QV* in *:II.* Join *PM.*

By Prop. 19, PI/ = ARM.

Also *PV=2Y.111.*

Therefore YM =2RY,

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and *APQM=2APRQ.*

Hence *APQV=4APRQ,*

and *APQq=8APRQ.*

Also, if RTV, the ordinate from *1?* to PV, he produced to **meet** the curve

again in r, *RTV=7•Fir,*

and the same proof shows that

*PQ(/•=8APrq.*

PROPOSITION 22

*If there be a series of areas A, B, C, I), • • • each of which is four times the next in order, and if the largest, A, be equal to the triangle PQq inscribed in a parabolic segment PQq and having the same base with it and equal height, then*

*(A+B+C-}-D-}- • • •)<(area of segment PQq).*

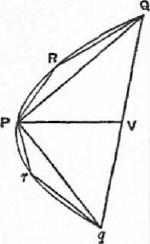
For, since *APQq=8APRQ=8APqr,* where *I?,* r are the vertices of the segments cut off by *PQ, Pq,* as in the last proposition,

PQq=4(APQR+APqr). Therefore, since APIQq= *A,*

*APQR+APqr=B•*

In like manner we prove that the triangles similarly in­scribed in the remaining segments are together equal to the area C, and so on.

Therefore *Ad-B+C±D-1- • • •* is equal to the area of a certain inscribed polygon, and is therefore less than the area of the segment.



PROPOSITION 23

*Given a series of areas A, B, C, D, • • •Z, of which A is*

*equal to four times the next in order, then A -1-B+C+ • • •+Z-1-V=I-A.* Take areas *b, c, d, • • •* such that *b= 3B,*

*c=1C,*

*d=* 1D, and so on. Then, since *b =1/3,*

and *B =1A,*

*B+b=i-A.*

Similarly *C-i-c=iB.*

*the greatest, and each is*

**A**

F

C

Therefore

*13-1-C-FD+ • • •±Z+b-1-c-i-d+ • • •+z=*

*EA-1-B±C+ • • --FY).*

But

*b+c-I-d+ • • • -FY —1(B+C-1-D+ • • •-i-Y).*

Therefore, by subtraction,

*B+C-I-D+ • • •-1--Z-Fz-=-1A*

or *A-I-B-1-C+ • • •-1-Z-HlyZ=4-A.*

QUADRATURE OF THE PARABOLA 135

PROPOSITION 24

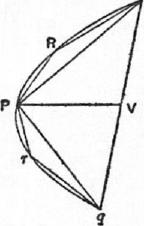
*Every segment bounded by a parabola and a chord Qq is equal to four-thirds of the triangle 'which has the same base as the segment and equal height.*

Suppose *K =tAPQq,*

where *P* is the vertex of the segment; and we have then; to prove that the area

of the segment is equal to *K.*

**Q** For, if the segment be not equal to *K,* it must either be  
greater or less.



I. Suppose the area of the segment greater than *K.*

If then we inscribe in the segments cut off by *PQ, Pq* triangles which have the same base and equal height, i.e. triangles with the same vertices R, **r** as those of the seg­ments, and if in the remaining segments we inscribe tri­angles in the same manner, and so on, we shall finally have segments remaining whose sum is less than the area by which the segment *PQq* exceeds *K.*

Therefore the polygon so formed must be greater than the area *K;* which is impossible, since [Prop. 23]

*A+B-1-C-1- • • •-1-Z <AA,*

where *A= APQq.*

Thus the area of the segment cannot be greater than *K.*

II. Suppose, if possible, that the area of the segment is less than *K.*

If then *APQq= A, B = 144, C =113,* and so on, until we arrive at an area *X*

such that *X* is less than the .difference between *K* and the segment, we have

*A+B-I-C-1- • • .+X-1-4X=4A* [Prop. 23]

*=K.*

Now, since *K* exceeds *A+B-I-C+ • • •+X* by an area less than X, and the

area of the segment by an area greater than *X,* it follows that

*A-1-B+C+ • • •-I-X>* (the segment);

which is impossible,.by Prop. 22 above.

Hence the segment is not less than *K.*

Thus, since the v•egment is neither greater nor less than *K,*

(area of segment *PQq) = K =1,APQq,*

ON FLOATING BODIES  
BOOK ONE

POSTULATE 1

"Let it be supposed that a fluid is of such a character that, its parts lying evenly and being continuous, that part which is thrust the less is driven along by that which is thrust the more; and that each of its parts is thrust by the fluid which is above it in a perpendicular direction if the fluid be sunk in any­thing and compressed by anything else."

**PROPOSITION 1**

*If a surface be cut by a plane always passing through* a *certain point, and if the section be always a circumference [of a circle] whose centre is the aforesaid point, the surface is that of a sphere.*

For, if not, there will be some two lines drawn from the point to the surface Which are not equal.

Suppose *0* to be the fixed point, and *A, B* to be two points on the surface such that *OA, OB* are unequal. Let the surface be cut by a plane passing through *0A, OB.* Then the section is, by hypothesis, a circle whose centre is *O.*

Thus *0A = 0B;* which is contrary to the assumption. Therefore the surface cannot but be a sphere.

**PROPOSITION** 2

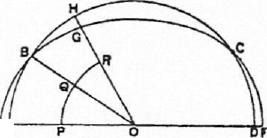
*The surface of* any *fluid at rest is the surface of a sphere whose centre is the same as that of the earth.*

Suppose the surface of the fluid cut by a plane through 0, the centre of the earth, in the curve *ABCD.*

*ABCD* shall be the circumference of a circle.

For, if not, some of the lines drawn from *0* to the curve will be unequal. Take one of them, *OB,* such that *OB* is greater than some of the lines from *0* to the curve and less than others. Draw a circle with *OB* as radius. Let it be *EBF,* which will therefore fall partly within and partly without the surface of the fluid.

Draw *OGH* making with *OB* an angle equal to the angle *EOB,* and meeting the surface in *H* and the circle in *G.* Draw also in the plane an arc of a circle *PQR* with centre 0 and within the fluid.



Then the parts of the fluid along *PQR* are uniform and continuous, and the part **E A** *PQ* is compressed by the part between it and *A B,* while the part *QR is* compressed by the part between *QR* and *BH.*

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Therefore the parts along ***PQ, QR*** will be unequally compressed, and the part which is compressed the less will be set in motion by that which is compres­sed the more.

Therefore there will not be rest; which is contrary to the hypothesis. Hence the section of the surface will be the circumference of a circle whose centre is 0; and so will all other sections by planes through 0.

Therefore the surface is that of a sphere with centre ***0.***

**PROPOSITION** 3

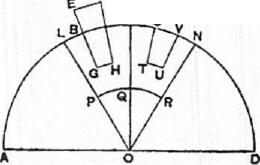
*Of solids those which, size for size, are of equal weight with a fluid will, if let down into the fluid, be immersed so that they do not project above the surface but do not sink lower.*

If possible, let a certain solid ***EFHG*** of equal weight, volume for volume, with the fluid remain immersed in it so that part of it, ***EBCF,*** projects above the surface.

Draw through ***0,*** the centre of the earth, and through: the solid a plane cut­ting the surface of the fluid in the circle ***ABCD.***

Conceive a pyramid with vertex ***0*** and base.a parallelogram .at the surface of the fluid, such that it includes the immersed portion of the solid. Let this pyramid be cut by the plane of ***A BCD*** in ***OL,***

***c M q OM.*** Also let a sphere within the fluid and be-  
low ***Gil*** be described with centre 0, and let the plane of ***A BCD*** cut this sphere in ***PQR.***



Conceive also another pyramid in the fluid with vertex ***0,*** continuous with the former pyramid and equal and similar to it. Let the pyramid so described be cut **in *OM, ON*** by the plane of ***ABCD.***

Lastly, let ***ST UV*** be a part of the fluid within the second pyramid equal and similar to the part ***BGHC*** of the solid, and let *SV* be at the surface of the fluid.

Then the pressures on ***PQ, QR*** are unequal, that on ***PQ*** being the greater. Hence the part at ***QR*** will be set in motion by that at ***PQ,*** and the fluid will not be at rest; which 'is contrary to the hypothesis.

Therefore the solid will not stand out above the surface.

Nor will it sink further, because all the parts of the fluid will be under the same pressure.

**PROPOSITION** 4

*A solid lighter than a fluid will, if immersed in it, not be completely submerged, but part of it will project above the :surface.*

In this case, after the manner of the previous proposition; we assume the solid, if possible, to be completely submerged and the fluid to be at rest. in that position, and.we conceive (1) a pyramid with its vertex at ***0,*** the centre of the earth, including the solid, (2) another pyramid continuous with the former and equal and similar to it, with the same vertex ***0,*** (3) a portion of the fluid within this latter pyramid equal to the immersed solid in the other pyramid, (4) a sphere with centre 0 whose surface is beloW the immersed solid and the part of the fluid in the second pyramid corresponding thereto. We suppose a plane to be drawn through the centre ***0*** cutting the surface of the fluid in the circle

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*ABC,* the solid in *S,* the first pyramid in *OA, OB,* the second pyramid in *OB,*

*OC,* the portion of the fluid in the second pyr. amid in *K,* and the inner sphere in *PQR.*

Then the pressures on the parts of the fluid at *PQ, QR* are unequal, since *S* is lighter than *K.* Hence there will not be rest; which is con­trary to the hypothesis.

Therefore the solid *S* cannot, in a condi­tion of rest, be completely submerged.

PnoposiTioN 5

*Any solid lighter than a fluid will, if placed in the fluid, be so far immersed that the weight of the solid will be equal to the weight of the fluid displaced.*

For let the solid be *EGHF,* and let *BGIIC* be the portion of it immersed when the fluid is at rest. As in Prop. 3, conceive a pyramid with vertex *0* in­cluding the solid, and another pyramid with the same vertex continuous with

the former and equal and similar to it. Sup­pose a portion of the fluid *STUV* at the base of the second pyramid to be equal and similar to the immersed portion of the solid; and let the construction be the same as in Prop. 3.

Then, since the pressure on the 'parts of the fluid at *PQ, QR* must be equal in order that the fluid may be at rest, it follows that

the weight of the portion *STUV* of the fluid must be equal to the weight of the solid *EGHF.* And the former is equal to the weight of the fluid displaced by the immersed portion of the solid *BGHC.*

PROPOSITION **6**

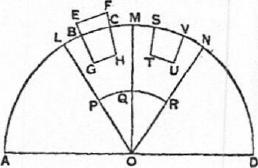
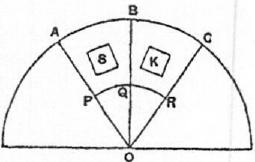
*If* a *solid lighter than a fluid be forcibly ,immersed in it, the solid will be drive\* upwards by a force equal to the difference between its weight and the weight of the fluid displaced.*

For let *A* be completely immersed in the fluid, and let *G* represent the weight of A, and *(G-I-H)* the weight of an equal volume of the fluid. Take a solid *D,* whose weight is *H* and add it to *A.* Then the weight of *(A + D)* is less than that of an equal volume of the fluid; and, if *(A +D)* is immersed in the fluid, it will

project so that its weight will be equal to the weight of the fluid displaced. But its weight is (G-F *H).*

Therefore the • weight of the fluid displaced is *(G+11),* and hence the volume of the fluid displaced is the vokime of the solid *A.* There will accordingly be rest with *A* immersed and *D* projecting:

Thus the weight of *D* balances the upward force exerted by the fluid on *A,* and therefore the latter force is equal to *H,* which is the difference between the weight of *A* and the weight of the fluid which *A* displaces.



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| G  4..1.11•10. |
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PROPOSITION 7

*A solid heavier than a fluid will, if placed in. it, descend to the bottom of the fluid, and the solid will, when weighed in the fluid, be lighter than its true weight by the weight of the fluid displaced.*

1. The first part of the proposition is obvious, since the part of the fluid under the solid Nvill be under greater pressure, and therefore the other parts will give way until the solid reaches the bottom.
2. Let *A* be a solid heavier than the same volume of the fluid, and let (G+11) represent its weight., while G represents the weight of the same volume of the fluid.

Take a solid *B* lighter than the same volume of the fluid, and such that the weight of *B* is G, while the weight of the same volume of the fluid is (G+*11).*

A

Let A and *B* be now combined into one solid

B

H and immersed. Then, since (A *+B)* Nvill be of the  
same weight as the same volume of fluid, both weights being equal to *(G+H)+G,* it follows that (A *+B)* will remain stationary in the fluid. Therefore the force which causes A by itself to sink must he equal to the upward force exerted by the fluid on *B* by itself. This latter is equal to the difference between (G+H) and [Prop. 6]. Hence A is depressed by a force equal to *II, i.e.* its weight in the fluid is *II,* or the difference between ((7-1-11) and G.

POSTUI-1TE

"Let it be granted that bodies which are forced upwards in a fluid are forced upwards along the perpendicular [to the surface] which passes through their centre of gravity."

PROPOSITION 8

*If a. solid in the form of a segment of a sphere, and of a substance lighter than a fluid, be immersed in it so that its base does not touch the surface, the solid will rest in. such a position that its axis is perpendicular to the surface; and, if the solid be forced into such* a *position that its base touches the fluid on one side and be then set free, it will not remain in that position but will return to the symmetrical position.*

PROPOSITION 9

*If a solid in the form of a segment of a sphere, and of a substance lighter than a fluid, be immersed in it so that its base is completely below the surface, the solid will rest in such* a *position that its nxis is perpendicular to the surface.*

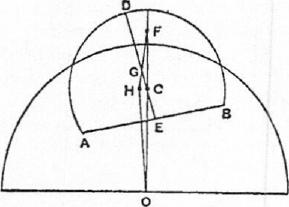
[The proof of this proposition has only survived in a mutilated form. It deals moreover with only one case out of three which are distinguished at the begin­ning, viz. that in which the segment is greater than a hemisphere. . . .]

Suppose, first, that the segment is greater than a hemisphere. Let it be cut by a plane through its axis and the centre of the earth; and, if possible, let it be at rest in the position shown in the figure, where *AB* is the intersection of

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the plane with the base of the segment, *DE* its axis, *C* the centre of the sphere of which the segment is a part, 0 the centre of the earth.

The centre of gravity of the portion of the segment outside the fluid, as *F,* lies on *OC* produced, its axis passing through *C.*



Let *G* be the centre of gravity of the segment. Join *FG,* and produce it to *II* so that

*FG : GH =* (volume of immersed portion) : (rest of solid).

Join *OH.*

Then the weight of the portion of the solid outside the fluid acts along *FO,* and the pressure of the fluid on the immersed portion along *011,* while the weight of the immersed portion acts along *110* and is by hypothesis less than the pressure of the fluid acting along *OH.*

Hence there will not be equilibrium, but the part of the segment towards A will ascend and the part towards *B* descend, until *DE* assumes a position per­pendicular to the surface of the fluid.

ON FLOATING BODIES  
BOOK TWO

PROPOSITION

*If* a *solid lighter than a fluid be at rest in it, the weight of the solid will be to that of  
the same volume of the fluid as the immersed portion of the solid is to the whole.*Let *(A +B)* be the solid, *B* the portion immersed in the fluid.

Let (C+D) be an equal volume of the fluid,

E *C* being equal in volume to *A* and *B* to D. Further suppose the line *E* to represent the weight of the solid *(A +B), (F +G)* to repre­sent the weight of (C+D), and *G* that of *D.*

**A**

F Then

weight of *(A +B) :* weight of *(C+D)*

0

*E : (F +G). (1)*And the weight of *(A + B)* is equal to the weight of a volume *B* of the fluid [I. 5], i.e. to the weight of *D.*

That is to say, *E = G.*

Hence, by (1),

weight of *(A +B) :* weight of *(C+D) = G : F+G*

*=D :C+D*

*=B:A+B.*

PROPOSITION 2

*If a right segment of a paraboloid of revolution whose axis is not greater than ip (where p is the principal parameter of the generating parabola), and whose specific gravity is less than that of a fluid, be placed in the fluid with its axis inclined to the vertical at any angle, but so that the base of the segment does not touch the surface of the fluid, the segment of the paraboloid will not* remain *in that position but will return to the position in which its axis is vertical.*

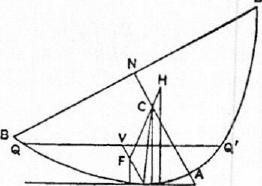
Let the axis of the segment of the paraboloid be *AN,* and through *AN* draw a plane perpendicular to the surface of the fluid. Let the plane intersect the paraboloid in the parabola *BAB',* the base of the segment of the paraboloid in *BB',* and the plane of the surface of the fluid in the chord QQ' of the parabola.

Then, since the axis *AN* is placed in a position not perpendicular to QQ', *BB'* **will** not be parallel to *QQ'.*

Draw the tangent *PT* to the parabola which is parallel to QQ', and let *P* be the point of contact.'

'The rest of the proof ... is given in brackets as supplied by Commandinus.

**141**



|  |  |
| --- | --- |
| **L PKM** | T |

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[From *1'* draw PI' parallel to AN meeting QQ' in V. Then *P1'* will be a diameter of the parabola, and also the axis of the portion of the paraboloid immersed in the fluid.

Let *C* he the centre of gravity of the pa-

raboloid *BAB',* and that of the portion immersed in the fluid. Join *FC* and produce it to *II* so that. *II* is the centre of gravity of the remaining portion of the paraboloid

above the surface.

Then, since A N = *C,*

and AN>b),

it follows that ilC>P

2'

Therefore, if *CP* he joined, the angle *CPT*

is acute. Hence, if *CK* be drawn perpendicular to *PT, K* will fall between *P* and *T.* And, if *FL, HM* be drawn parallel to *CK* to meet *P7',* they Neill each be perpendicular to the surface of the fluid.

Now the force acting on the immersed portion of the segment of the parabo­loid **will** act upwards along *LF,* while the weight. of t he portion outside the fluid will act downwards along *JIM.*

Therefore there will not be equilibrium, but the segment will turn so that *B* will rise and *B'* will fall, until *AN* takes the vertical position.]

PROPOSITION 3

*If a right segment of a paraboloid of revolution whose axis is not greater than ip (where p is the parameter), and whose specific gravity is less than that of a fluid, be placed in the fluid with its axis inclined at any angle to the vertical, but so that its base is entirely submerged, the solid will not remain in that position but will return to the position in which the axis is vertical.*

Let the axis of the paraboloid be *AN,* and through AN draw a plane perpen­dicular to the surface of the fluid intersecting the paraboloid in the parabola *BAB',* the base of the segment in *BNB',* and the plane of the surface of the

fluid in the chord *QQ'* of the parabola.

Then, since *AN,* as placed, is not perpen­dicular to the surface of the fluid, QQ' and *BB'* will not be parallel.

Draw *PT* parallel to *QQ'* and touching the

parabola at *P.* Let *PT* meet NA produced in tzt

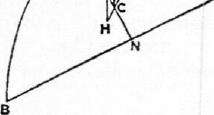
T. Draw the diameter *PV* bisecting *QQ'* in V. PV is then the axis of the portion of the paraboloid above the surface of the fluid.

Let *C* be the centre of gravity of the whole segment of the paraboloid, *F* that of the por­tion above the surface. Join *FC* and produce

it. to *II* so that H is the centre of gravity of the immersed portion.

Then, since A C> -2, the angle *CPT* is an acute angle, as in the last prop­osition.

**MK PL**



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Hence, if *CK* be drawn perpendicular to *PT, K* will fall between *P* and *'P.* Also, if *JIM, FL* be drawn parallel to *CK,* they will be perpendicular to the surface of the fluid.

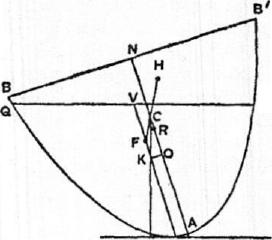
And the force acting on the submerged portion will act upwards along *HM,* while the weight of the rest will act downwards along *LF* produced.

Thus the paraboloid will turn until it takes the position in which *AN* is vertical.

**PROPOSITION** 4

Given a *right segment* of a paraboloid of revolution *whose axis AN is greater than ip (where p is the parameter), and whose specific gravity is less than that of a fluid but bears to it a ratio not less than (A N —1-p)* **2 :** *A N2, if the segment of the parabo­loid be placed in the fluid with its axis at any inclination to the vertical, but so that its base does not touch the surface of the fluid, it will not remain in that position but will return to the position in which its axis is vertical.*

Let the axis of the segment of the paraboloid be *AN,* and let a plane be drawn through *AN* perpendicular to the surface of the fluid and intersecting the segment in the parabola *BAB',* the base of the segment in *BB',* and the surface of the fluid in the chord *QQ'* of the parabola.



Then *AN,* as placed, will not be perpendic­ular to *QQ'.*

**Q'** Draw *PT* parallel to QQ' and touching the  
parabola at *P.* Draw the diameter *PV* bisect­ing QQ' in V. Thus *PV* will be the axis of the submerged portion of the 'solid.

C be the centre of gravity of the whole solid, *F* that of the immersed portion. Join *FC* and produce it to ***H*** so that *H* is the centre of gravity of the remaining portion. *AN =4AC,*

*AN>ip,*

P

*Now,* since  
and

it follows that

*AC >* 2-\*

Measure *CO* along *CA* equal to -2' and *OR* along *OC* equal to *-I-AO.*

Then, since *AN =IAC,*

and *AR*

we have, by subtraction, *NR WC.*

That is, *AN—AR=WC*

*=iP,*

or *AR= (AN —* tp).

Thus ***(AN*** ---ip)2 *AN2 =AR2* : *AN2,*

and therefore the ratio of the specific gravity of the solid to that of the fluid is, by the enunciation, not less than the ratio *AR2 A*N2.

But, by Prop. 1, the former ratio is equal to the ratio of the immersed por­tion to the whole solid, i.e. to the ratio *PV2* : *AN2 [On Conoids and Spheroids,* Prop. 24].

Hence *PV2* : AN2<AR2 : AN2,

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or PIT<AR.

It follows that *PF(=3PV)<1AR*

*<AO.*

If, therefore, *OK* be drawn from *0* perpendicular to *OA,* it will meet *PF* be­tween *P* and *F.*

Also, if *CK* be joined, the triangle *KCO* is equal and similar to the triangle formed by the normal, the subnormal and the ordinate at *P* (since *CO* = 1p or the subnormal, and *KO* is equal to the ordinate).

Therefore *CK* is parallel to the normal at P, and therefore perpendicular to the tangent at *P* and to the surface of the fluid.

hence, if parallels to *CK* be drawn through *F, H,* they will be perpendicular to the surface of the fluid, and the force acting on the submerged portion of the solid will act upwards along the former, while the weight of the other portion will act downwards along the latter.

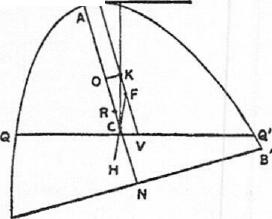
Therefore the solid will not remain in its position but will turn until *AN* assumes a vertical position.

**PROPOSITION** 5

*Given a right segment of* a *paraboloid of revolution such that its axis AN is greater than ip (where p is the parameter), and its specific gravity is less than that of a fluid but in a ratio to it not greater than the ratio {AN2 — (AN —* ip)2} *AN2, if the segment be placed in the fluid with its axis inclined at any angle to the vertical, but so that its base is completely submerged, it will not remain in that position but will return to the position in which AN is vertical.*

Let a plane be drawn through *AN,* as placed, perpendicular to the surface of the fluid and cutting the segment of the paraboloid in the parabola *BAB',*

the base of the segment in *BB',* and the **T P** plane of the surface of the fluid in the chord *QQ'* of the parabola.



Draw the tangent *.PT* parallel to *QQ',* and the diameter *PV,* bisecting QQ', will accordingly be the axis of the portion of the paraboloid above the surface of the fluid.

Let *F* be the centre of gravity of the por­tion above the surface, *C* that of the whole solid, and produce *FC* to *H,* the centre of

gravity of the immersed portion. a

As in the last proposition, *Aci>1-,* and we measure *CO* along *CA* equal to *I*

and *OR* along *OC* equal to *4.A0.*

Then *AN =MC,* and *AR=* I-40;

and we derive, as before, AR *= (AN —1p).*

Now, by hypothesis,

(spec. gravity of solid) : (spec. gravity of fluid)

> tAN2— (AN-179)2i AN2

> (AN2 *— A R2) : A N2 .*

Therefore

(portion submerged) : (whole solid)

> (AN= — A R2) : AN2,

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and (whole solid) : (portion above surface)

> A" : A /V.

Thus *AN2 1.72> A N2* A *R2,*

whence *PV* <A R,

and *PF<AR*

SAO.

Therefore, if a perpendicular to AC be drawn from *0,* it will meet. *PF* in

some point ***K*** between *P* and F.

And, since *CO =D), CK* **Nv i 11** be perpendicular to *PT,* as in the last prop-

osition.

Now the force acting on the submerged portion of the solid will act upwards

through ***If,*** and the weight of the other portion downwards through *F, in* direc-

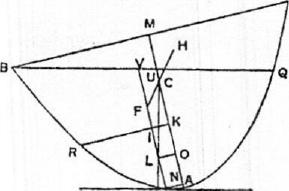
tions parallel in both cases to *CK;* whence the proposition follows.

**PROPOSITION** 6

*If a right segment. of a paraboloid lighter than a fluid be such that its axis AM is greater than p, but AM :1-p <15* : 4, *and if the segment be placed in. the fluid with its axis so inclined to the vertical that its base touches the fluid, it will never remain in such a position that the base touches the surface in one point only.*

Suppose the segment of the paraboloid to be placed in the position described, and let the plane through the axis *AM* perpendicular to the surface of the fluid intersect the segment of the paraboloid in the parabolic segment *BA B'* and the plane of the surface of the fluid in *BQ.*

Take *C* on AM such that *AC =2CM*



**B'** (or so that *C* is the centre of gravity of  
the segment of the paraboloid), and measure *CK* along *CA* such that

*AM :CK=15:* 4.

Thus AM : *CK>AM* by hypo-  
thesis; therefore *CK*

Measure *CO* along CA equal to Also draw *KR* perpendicular. to' *AC* meeting the parabola in *R.*

Draw the tangent *PT* parallel to *BQ,* and through *P* draw the diameter *PV* bisecting BQ in *V* and meeting *KR* in I.

Then *PV :PI*or> *KM : AK,*

*"for this is proved."*

*And CK=AAM =MC;*

whence *AK=AC—CK=RAC=MM.*

Thus *KM = MM.*

Therefore *KM = 4A K.*

It follows that ***PV07>IPI,***

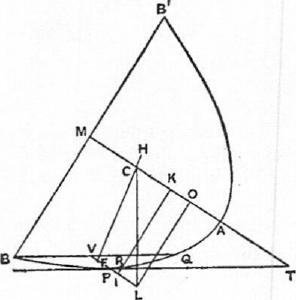
so that Plor<2117.

Let ***F* be the centre of gravity of the immersed portion of the paraboloid,** so **that *PF = 2FV.*** Produce *FC* to ***H,*** the centre of gravity of the portion above the surface.

Draw ***OL*** perpendicular to *PV.*

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Then, since ***CO=ip, CL*** must be per­pendicular to ***PT*** and therefore to the surface of the fluid.



And the forces acting on the immersed portion of the paraboloid and the portion above the surface act respectively upwards and downwards along lines through ***F*** and ***H*** parallel to ***CL.***

Hence the paraboloid cannot remain in the position in which ***B*** just touches the surface, but must turn in the direction of increasing the angle ***PTM.***

The proof is the same in the case where the point ***I*** is not on ***VP*** but on ***VP*** pro­duced, as in the second figure.

**PROPOSITION** 7

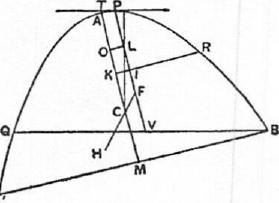
*Given a right segment of a paraboloid of revolution lighter than a fluid and such*

*that its axis AM is greater than ?fp, but.A : !,p, <15* : 4, *if the segment be placed  
in the fluid so that its base is entirely submerged, it will never rest in such a posi­tion that the base touches the surface of the fluid at one point only.*

Suppose the solid so placed that one point of the base only *(B)* touches the surface of the fluid. Let the plane through ***B*** arid the axis *AM* cut the solid in the parabolic segment. ***BAB'*** and the plane of the surface of the fluid in the chord ***BQ*** of the parabola.

Let *C* be the centre of gravity of the segment., so that A *C=* 2C3/; and meas­ure *CK* along ***CA*** such that

***AM :CK =15*** : 4.



It follows that ***CK<lp.***

Measure ***CO*** along *CA* equal to Draw ***KR*** perpendicular to ***AM.*** meeting the parabola in ***R.***

Let ***PT,*** touching at ***P,*** be the tangent to the parabola which is parallel to ***BQ,*** and ***PV*** the diameter bisecting ***BQ,*** the axis of the portion of the paraboloid above the surface.

Then, as in the last proposition, we prove that

***PI"***

* ***or> -***

and ***PI =*** 2/17.

or<

Let ***F*** be the centre of gravity of the portion of the solid above the surface; join *FC* and produce it to ***H,*** the centre of gravity of the portion submerged.

Draw ***OL*** perpendicular to ***PV,*** and, as before, since ***CO = i p,*** *CL is* perpen­dicular to the tangent ***PT.*** And the lines through ***H, F*** parallel to *CL* are per­pendicular to the surface of the fluid; thus the proposition is established as before.

The proof is the same if the point ***I*** is not on *VP* but on ***VP*** produced.

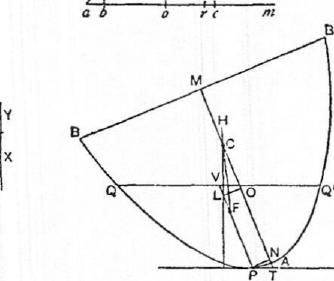
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PROPOSITION S

*Given a solid in the form of a right segment of a paraboloid of revolution whose axis*

AM *is greater than 4p, but such that AM < 15* : 4, and *whose specific gravity  
bears to that of a fluid a ratio less than* (AM — p)2 : A M2, *then, if the solid be placed in the fluid so that its base does not touch the fluid and its axis is inclined at an angle to the vertical, the solid will not return to the position. in which its axis is vertical and will not remain in any position except that in which its axis makes with the surface of the fluid a certain angle to be described.*

Let *am* be taken equal to the axis AM, and let *c* be a point ,on *am* such that *ac = 2cm.* Measure *co* along *ca* equal to 3p, and *or* along *oc* equal to 2a°.



***4***

***0***

*01*

I,

***C***

P

Let X+ Y be a straight line such t hat

(spec. gr. of solid) : ([spec. gr](http://spec.gr). of

fluid) = *(X+* : am2, (a)

and suppose *X* =2Y.

Now ar=4ao= a(iorn— 1p)

*=am* —*41)*

*= AM —*

Therefore, by hypothesis,

(X+ Y)2 : *ant" <ar' : am',*

whence *(X+* Y) *<ar,* and therefore

*X <ao.*

Measure *ob* along *oa* equal to X, and draw *bd* perpendicular to *ab* and of such

length that *bd2=lco • ab. ((3)  
Join* ad.

Now let the solid be placed in the fluid with its axis AM inclined at an angle to the vertical. 'Ishrough *AM* draw a plane perpendicular to the surface of the fluid, and let this plane cut the paraboloid in the parabola *BAB'* and the plane of the surface of the fluid in the chord *QQ'* of the parabola.

Draw the tangent *PT* parallel to QQ', touching at ***I',*** and let *PV* be the diameter bisecting QQ' in V (or the axis of the immersed portion of the solid), and PN the ordinate from *P.*

Measure AO along AM equal to *ao,* and *OC* along OM equal to *oc,* and draw OL perpendicular to *PV.*

I. Suppose the angle *07'P* greater than the angle *dab.*

*Thus PA" : NT2>db2 :* ba2.

But PN2 : *N T2= p :* 4A N

*=co : NT,*

and *db2* : *ba2= leo : ab,* by (a).

Therefore *NT <* 2ab,

or AiV<ab,

whence *NO>bo* (since ao =AO)

*>X.*

Now (X+ 11)" : ant' = ([spec. gr](http://spec.gr). of solid) : (spec. gr. of fluid) = (portion immersed) : (rest of solid)

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= PV2 :

so that *X+Y=PV.*

But *PL(=NO)> X*

*>3(X+* Y), since *X=* 2Y,

>IPV,

**or** *PV <iPL,*

and therefore *PL> 2LV.*

Take a point *F* on *PV* so that *PP =2FV ,* i.e. so that *F* is the centre of grav-

ity of the immersed portion of the solid.

Also *AC=* ac= lam= 3AM, and therefore *C* is the centre of gravity of the

whole solid.

Join *FC* and produce it to *//,* the centre of gravity of the portion of the solid

above the surface.

Now, since *CO=ip, CL* is perpendicular to the surface of the fluid;

therefore so are the parallels to *CL* through and *II.* But the force on the

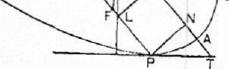
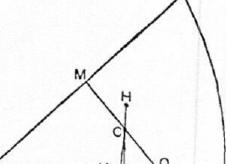
immersed portion acts upwards through *F* and that on the rest of the solid

downwards through II.

Therefore the solid will not rest but turn in

the direction of diminishing the angle *3ITP.*

1. Suppose the angle *OTP* less than the angle *dab.* In this case, we shall have, in­stead of the above results, the following,



*AN>ab,*

NO <X.

Also P > *iPL,*

and therefore *PL <2LT' .*

Make *PF* equal to 2FV, so that is the centre of gravity of the immersed portion.

And, proceeding as before, we prove in this case that the solid will turn in the di­rection of *increasing* the angle *MTP.*

1. When the angle *MTP* is equal to the angle *dab,* equalities replace inequalities in the results obtained, and *L* is itself the centre of gravity of the immersed portion. Thus all the forces act in one straight line, the perpendicu­lar *CL;* therefore there is equilibrium, and the solid will rest in the position described.

PROPOSMON 9

*Given a solid in the form of a right segment of a paraboloid of revolution whose axis AM is greater than p, but such that AM : ?ip<* 15 : 4, *and whose specific gravity bears to that of a fluid a ratio greater than {AM2— (AM —;p)2}* : *AM2, then, if the solid be placed in the fluid with its axis inclined at an angle to the vertical but so that its base is entirely below the surface, the solid will not return to the position in which its axis is vertical and will not remain in any position except that in which its axis makes with the surface of the fluid an angle equal to that described in the last proposition.*

Take am equal to *AM,* and take *c* on am such that *ac =2cm.* Measure *co* along *ca* equal to ip, and *or* along *ac* such that ar= *gao.*

Let *X+ Y* he such a line that

**ON FLOATING BODIES II 149**

**(spec. gr. of solid) : (spec. gr. of fluid) = lam2 — (X+ Y)9 : am2, and suppose *X = 2Y.***

**Now ar = lap**

|  |  |  |  |
| --- | --- | --- | --- |
| d  A  **a*b*** | 0 | II  re | *m* |

T



**=1(lam*—p)***

***=AM—b.***

**Therefore, by hypothesis,**

***am2—ar2 :am' <* {amt — (X+ Y)9 : am2,**

**,. whence X-F-Y<ar,**

**and therefore**

**Make *ob* (measured along *oa)* equal**

**to *X,* and draw *bd* perpendicular to *ba***

***X* <ao.**

**and of such length that**

***bd2= lc°* • ab.**

**Join *ad.***

**Now suppose the solid placed as in the figure with its axis *AM* inclined to the vertical. Let the plane through *AM* perpendicular to the surface of the fluid cut the solid in the parabola *BAB'* and the surface of the fluid in *QQ'.* Let *PT* be the tangent parallel to *QQ', PV* the diameter bisecting QQ' or) the axis of the portion of the paraboloid above the surface), *PN* the ordinate from *P.***

**I. Suppose the angle *MTP* greater than the angle *dab.* Let *AM* be cut as before in *C* and *0* so that *AC = 2CM , OC =ip,* and accordingly *AM, am* are equally divided. Draw OL perpendicular to *PV.***

**Then, we have, as in the last proposition,**

***PN2* : *NT2>db2 : ba2,***

**whence *co :NT>lco :* ab,**

**and therefore *AN* <ab.**

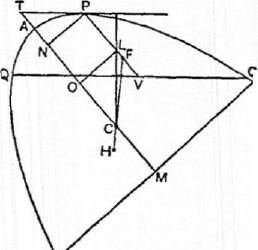
**It follows that *NO>bo***

***>X.***

**Again, since the specific gravity of the solid is to that of the fluid as the**

**immersed portion of the solid to the whole,**

***AM2—* (X+ Y)2 : *AM2=AM2—PV2 AM2,***



**or (X+ Y)2 : *AM2=PV2* : AM2.**

**That is, *X+Y=PV.***

**And *PL* (or *NO)> X***

**8' > *§PV,***

**so that *PL>2LV.***

**Take *F* on *PV* so that *PF =2FV.* Then *F* is the centre of gravity of the portion of the solid above the surface.**

**Also *C* is the centre of gravity of the whole solid. Join *FC* and produce it to *H,* the cen­tre of gravity of the immersed portion.**

**Then, since *CO =* ip, *CL* is perpendicular to *PT* and to the surface of the fluid; and**

**the force acting on the immersed portion of the solid acts upwards along the**

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Parallel to *CL* through *H,* while the weight. of the rest of the solid acts down­wards along the parallel to *CL* through *F.*

Hence the solid will not rest but turn in the direction of diminishing the angle *MTP.*

*II.* Exactly as in the last proposition, we prove that, if the angle *MTP* be less than the angle *dab,* the solid will not remain in its position but will turn in the direction of increasing the angle *MTP.*

I If. If the angle *MTP* is equal to the angle *dab,* the solid will rest. in that position, because *L* and *F* will coincide, and all the forces will act along the one line *CL.*

PROPOSITION 10

*Given a solid in the form of a right segment of a paraboloid of revolution in which the axis AM is of a length such that AM :* ip> 15 : 4, *and supposing the solid placed in* a *fluid of greater specific gravity so that its base is entirely above the surface of the fluid, to investigate the positions of rest.*

(PRELIMINARY)

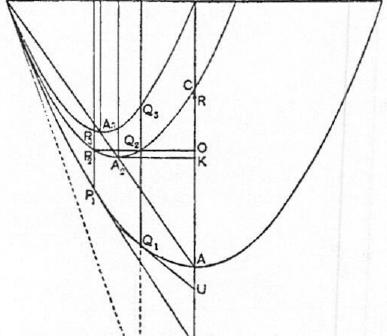
Suppose the segment of the paraboloid to be cut by a plane through its axis *AM* in the parabolic segment BA Bi of which *BB1* is the base.

Divide *AM* at *C* so that *AC=* 2CM, and measure *CK* along *CA* so that  
AM : *CK =15* : 4,

(a)

whence, by the hypothesis, *CK>ip.*

Suppose *CO* measured along 8 M-M: D  
*CA* equal to 2 p*,* and take a point *1?* on *AM* such that



*MR = NCO.*

'Thus *AR= A — MR*

*=3-(AC—CO)*

*=3Ao.*

Join *BA,* draw *KA2* perpendicu­lar to *AM* meeting *BA* in Az, bisectBA in A3, and draw A2312, A 3M3 parallel to *AM* meeting *BM* in *M2,* M3 respectively.

On *A 2M2, A 3M3* as axes de­scribe parabolic segments simi­lar to the segment *BA B1.* (It fol­lows, by similar triangles, that *BM* will be the base of the seg-

ment whose axis is A 33/3 and ,E

RB2 the base of that whose axis

is A2M2, where *BB2=2BM2.)*

The parabola *BA2R2* will then

pass through *C.*

[For

*13M2:M2M=BM2:A2K*

*=KM :AK*

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*=CM+CK : AC—CK*

*= (§4-is)A ji (3* — "YAM

=9:6 *(i3)  
=MA :AC.*

Thus *C* is seen to be on the parabola BA2B2 by the converse of Prop. 4 of the *Quadrature of the Parabola.]*

Also, if a perpendicular to AM be drawn from 0, it will meet the parabola *BA* 2B2 in two points, as Q2,. '**P2.** Let *Q1Q2Q3D* be drawn through Q2 parallel to *AM* meeting the parabolas *BA Bi, BA 3M* respectively in Q1, Q3 and *BM* in *D;* and let PIP2P3 be the corresponding parallel to *A M* through **P2.** Let the tan­gents to the outer parabola at PI, Q1 meet *MA* produced in T1, *U* respectively.

Then, since the three parabolic segments are similar and similarly situated, with their bases in the same straight line and having one common extremity, and since Q1Q2Q3D is a diameter common to all three segments, it follows that

Q1(22 : Q2(23= (B2/31 : *BIB)* • (B.1/ : MB2).

Now *B2131 : 13113=MM2: BM* (dividing by 2)

=2 : 5, by means of (t3) above.

And *BM :MB2=11:11 :* **(2BM2** *— BM)*

|  |  |  |
| --- | --- | --- |
|  | =5 : (6-5),  =5:1. | by means of *(0),* |
| It follows that or  Similarly Also, since | : (22(23= 2 : 1, *Qd22=2(22Q3.1 P iP2= 2P 2P3.*  *MR= WO =*  *AR=AM—MR = A M —ip.* |  |

*(ENLINTIATIoN)*

*If the segment of the paraboloid be placed in the fluid with its base entirely above*

*the surface, then*

*(I.) if*

*(spec. gr. of solid) : (spec. gr. of fluid)<A R2 : AM2*

**[ <(A***M* 1p)2 : *AM2],*

*the solid will rest in the position in which its axis AM is.vertical;*

*(11.) if*

*(spec. yr. of `solid) : (spec. gr. of fluid) < A R2 : AM2*

*but>Q1Q32 : AM2,*

*the solid will not rest with its base touching the surface of the fluid in one pint*

*only, but in such a position that its* bate *doss .zot touch the surface at any point*

*and its axis makes with the 'surface* an *angle\* greater than U;*

*("II. a) if*

*(spec. gr. of solid) : (spec. yr. of fluid)=Q1Q32 AM2,*

*the solid will rest and remain in the position in which the base touches the surface*

*of the fluid at one point only and the axis Makes-with,the surface an angle equal*

*to U;*

(III. b) *if*

*(spec. gr. of solid) : (spec. gr. of fluid) =* 1)11)32 *A M2,*

*the solid will rest with its base touching the surface of the fluid at one point only*

*and with its axis inclined to the surface at* **ati** *angle equal to T1;*

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1. *if*

*(*[*spec. gr*](http://spec.gr)*. of solid) : (spec. gr. of fluid)> PiP32 : AM2*

*but <Q1Q32 : AM2,*

*the solid will rest and remain in a position with its base more submerged;*

1. *if*

*(spec. gr. of solid) : (spec. gr. of fluid) <P1P32 : AM2,*

*the solid will rest in* a *position in which its axis is inclined to the surface of the*

*fluid at an angle less than T1, but so that the base does not even touch the surface*

*at one point.*

(PROOF)

1. Since *AM >* ip, and

([spec. gr](http://spec.gr). of solid) : ([spec. gr](http://spec.gr). of fluid) < *(A AI* : A M2,

it follows, by Prop. 4, that the solid will be in stable equilibrium with its axis

vertical.

1. In this case

(spec. gr. of solid) : ([spec. gr](http://spec.gr). of fluid) *<A R2 : AM2*

*but > QIQ 32 : A 112.*

|  |  |
| --- | --- |
|  |  |
| 1 |
|  |

Suppose the ratio of the specific gravities to be equal to /2 : A M2, so that

*l<AR* but >Q1(23.

Place P'V' between the two parabolas *BABI, BP3 Q3M* equal to *land* paral-

lel to AM; and let *P'V'* meet the intermediate parabola in *P.*

Then, by the same proof as before, we obtain

*P'F' =2F'V'.*

Let *P'T',* the tangent at *P'* to the outer parabola, meet MA in 7", and let

*P'N'* be the ordinate at *P'.*

.Join *BV'* and produce it to meet the outer parabola in Q'. Let OQ21'2 meet

PT' in *I.*

Now, since, in two similar and similarly situated parabolic segments with

bases *BM, BBi* in the same straight line, *BV', BQ'* are drawn making the same

angle with the bases,

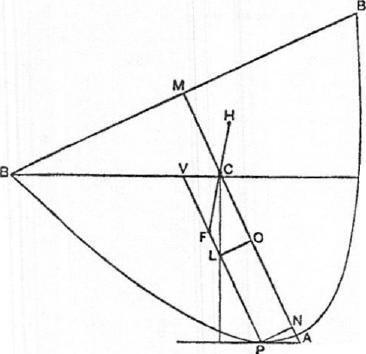
*By' : BQ' =BM : BBI*

*=1 :*2,

so that *BV' = V'Q'.*

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Suppose the segment of the paraboloid placed in the fluid, as described, with its axis inclined at an angle to the vertical, and with its base touching the surface at one point *B* only. Let the solid be cut by a plane through the axis and perpendicu­lar to the surface of the fluid, and let the plane intersect the solid in the parabolic segment *BAB'* and the plane of the .sur­face of the fluid in *BQ.*



Take the points *C, 0* on *AM* as before described. Draw the tan­gent parallel to *BQ* touching the parabola in *P* and meeting AM in *T;* and let *PV* be the diameter bisecting *BQ* (i.e. the axis of the

T immersed portion of the solid).

Then /2 : *AM2* **=** ([spec. gr](http://spec.gr). of solid) : ([spec. gr](http://spec.gr). of fluid) = (portion immersed) : (whole solid) *=PV2* : AM**2,**

whence *P'V'=/=Pl7.*

Thus the segments in the two figures, namely *BP'Q', BPQ,* are equal and

similar.

Therefore L *PTN = L P' 7"N'.*

Also *AT = AT', AN =AN', PN =P'N'.*

Now, in the first figure, *P'I <2117'.*

Therefore, if *OL* be perpendicular to *PV* in the second figure,

*PL<*2L*V.*

Take *F* on *LV* so that *PF=2FV,* i.e. so that *F* is the centre of gravity of the immersed portion of the solid. And *C* is the centre of gravity of the whole solid. Join *FC* and produce it to *H,* the centre of gravity of the portion above the surf ace.

Now, since *CO = 2 p, CL* is perpendicular to the tangent at *P* and to the surface of the fluid. Thus, as before, we prove that the solid will not rest with *B* touching the surface, but will turn in the direction of increasing the angle *PTN.*

Hence, in the position of rest., the **axis** AM must make with the surface of the fluid an angle greater than the angle *U* which the tangent at Q1 makes with *AM.*

*(III.* a) In this case

([spec. gr](http://spec.gr). of solid) : ([spec. gr](http://spec.gr). of fluid) *(2=* ,Q32 *AM2.* •

•

Let the segment of the paraboloid be placed in the fluid so that its base nowhere touches the surface of the fluid, and its axis is inclined at an angle to the vertical.

Let the plane through *AM* perpendicular to the surface of the fluid cut the paraboloid in the parabola *BAB'* and the plane of the surface of the fluid in *QQ'.* Let *PT* be the tangent parallel to QQ', *PV* the diameter bisecting *QQ', PN* the ordinate at *P.*

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Divide *AM as* before at *C, 0.*

In the other figure let Q1N'. be the ordinate at Qi. Join *BQ3* and produce it

to meet the outer parabola in *q.* Then *BQ3=Q3q,* and the tangent QiU is

parallel to *Bq.* Now

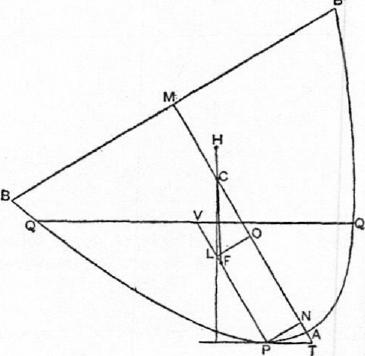
Q1(232 : *A1112* = ([spec. gr](http://spec.gr). of solid) : ([spec. gr](http://spec.gr). of fluid)

= (portion immersed) : (whole solid)

=PV2 : AM2.

Therefore QiQa=PV; and the

segments *QPQ', BQlq* of the



paraboloid are equal in volume.

And the base of one passes

through *B,* while the base of the

other passes through•Q, a point

nearer to A than *B* is.

It follows that the angle be-

tween QQ' and *BB'* is less than

the angle *BiBq.*

Therefore

*U< ZPTN,*

whence *AN' > AN,*

and therefore

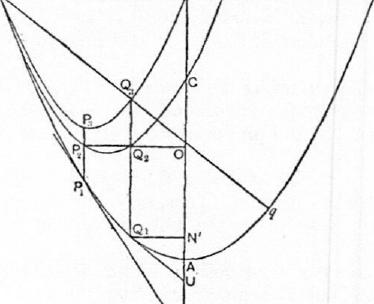
*N'0(or Q1Q2) <PL,*

where *OL* is perpendicular to

*PV.*

It follows, since Q1Q2 = 2Q2Q3,

that M B**,** *PL>2LV.*



Therefore *F,* the centre of gravity of the immersed portion of the solid, is between *P* and *L,* while, as before, *CL* is per­pendicular to the surface of the fluid.

Producing *FC* to *H,* the cen­tre of gravity of the portion of the. solid above the surface, we see that the solid must turn in the direction of diminishing the angle *PTN* until one point *B* of the base just touches the surface of the fluid. When this is the case, we shall

have a segment *BPQ* equal and similar to the segment *BQiq,* the angle *PTN* will be equal to the angle *U,* and *AN* will be equal to *AN'.*

Hence in this case *PL = 2LV,* and *F,* L coincide, so that *F, C, H* are all in one vertical straight line.

Thus the paraboloid will remain in, the position in which one point *B* of the base touches the surface of the fluid, and the axis makes with the surface an angle equal to *U.*

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(III. b) In the case where

([spec. gr](http://spec.gr). of solid) : ([spec. gr](http://spec.gr). of fluid) = PiP32 *AM2,* we can prove in the same way that, if the solid be placed in the fluid so that its axis is inclined to the vertical and its base does not anywhere touch the surface of the fluid, the solid will take up and rest in the position in which one point only of the base touches the surface, and the axis is inclined to it at an angle

equal to (in the figure on p. 552).

(IV.) In this case

(spec. gr. of solid) : ([spec. gr](http://spec.gr). of fluid) > PiP32 AM2

but <Q1(232 : *AM2.*

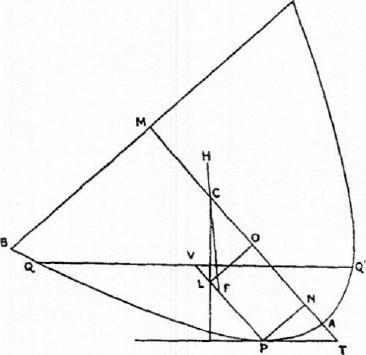
Suppose the ratio to be equal to /2 : AM2, so that l is greater than PiP3 but

less than (21(23.

|  |  |
| --- | --- |
|  | Place *P'V'* between the pa­rabolas BPal, *BP3* Q3 so that *P'V'* is equal to *1* and parallel to AM, and let *P'V'* meet the intermediate parabola in *F'* and *0Q2P2* in I.  Join *BV'* and produce it to meet the outer parabola in *q.* Then, as before, *BY' =*  and accordingly the tangent P'T' *P'* is parallel to *Bq.* Let /FA" be the ordinate of *P'.*  1. Now let the segment be placed in the fluid, *first,* with its axis so inclined to the ver­tical that its base does not |

anywhere touch the surface of the fluid.

Let the plane through *AM* perpendicular to the surface of the fluid cut the paraboloid in the parabola *BAB'* and the plane of the surface of the fluid in (2(2'. Let *PT* be the tangent par­allel to *QQ', PV* the diameter bisecting QQ'. Divide *AM at C, 0* as before, and draw *0/,* per­pendicular to *PV.*



Then, as before, we have PT' *=1= PP.*

Thus the segments *BP' q, QPQ'* of the paraboloid are equal in vol­ume; and it follows that the angle between *QQ'* and *BB'* is less than the angle *BiBq.*

Therefore

*L 1"1"A" < L PTN,* and hence A /V> A N,

so that NO *> N'0,*

*i.e. PL>P1*

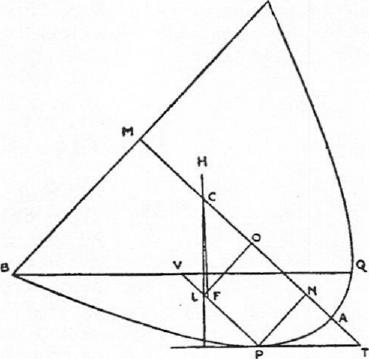
*> P'F', a fortiori.*

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Thus *PL>2LV,* so that *F,* the centre of gravity of the immersed portion of the solid, is between L and *P,* while *CL* is perpendicular to the surface of the

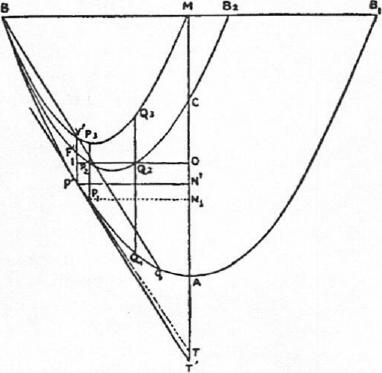
fluid.

s'



If then we produce *FC* to *H,* the centre of gravity of the portion of the solid above the surface, we prove that the solid will not rest but turn in the direc­tion of diminishing the angle *PTN.*

2. Next let the paraboloid be so placed in the fluid that its base touches the surface of the fluid at one point *B* only, and let the construction proceed as before.



Then *PV =P'V',* and the segments *BPQ, BP'q* are equal and similar, so that *PTN = L P'T'N'.*

It follows that *AN = AN', NO = N'0,*

and therefore *P'I = PL,*

whence *PL> 2LV.*

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Thus *F'* again lies between *P* and *L,* and, as before, the paraboloid will turn in the direction of diminishing the angle *PTN,* i.e. so that the base will be more submerged.

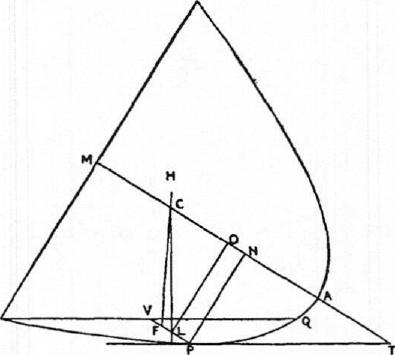
(V.) In this case

([spec. gr](http://spec.gr). of solid) : ([spec. gr](http://spec.gr). of fluid) <P1P32 : *AM2.*

If then the ratio is equal to /2 : *All12,1<P1133.* Place *P'V'* between the parabo­las BP1Q1 and *BP3Q3* equal in length to *1* and parallel to *AM.* Let *P'V'* meet the intermediate parabola in *F'* and *OP2* in *I.*

Join *BV'* and produce it to meet the outer parabola in *q.* Then, as before, *BV'=V'q,* and the tangent *P'T'* is parallel to *Bq.*

1. Let the paraboloid be so placed in the fluid that its base touches the sur­face at one point only.



Let the plane through *AM* perpendicular to the surface of the fluid cut the

paraboloid in the parabolic section *BAB'* and the plane of the surface of the

fluid in *BQ.*

Making the usual construction, we find

*PV=1=P'T",*

and the segments *BPQ, BPiq* are equal and similar.

Therefore L *PTN = L P'T'N',*

and *AN =AN', N'0=NO.*

Therefore *PL = P' I,*

whence it follows that *PL<2LV.*

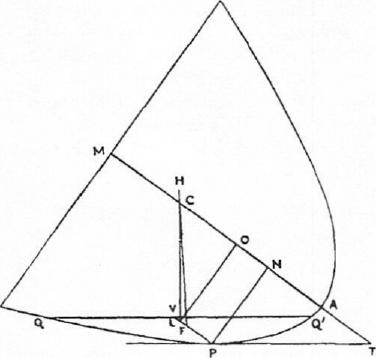
Thus *F,* the centre of gravity of the immersed portion of the solid, lies between *L* and V, while *CL* is perpendicular to the surface of the fluid.

Producing *FC* to *II,* the centre of gravity of the portion above the surface, we prove, as usual, that there will not be rest, but the solid will turn in the direction of increasing the angle *PTN,* so that the base will not anywhere touch the surface.

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2. The solid will however rest in a position where its axis makes with the surface of the fluid an angle less than T1.

**8'**



For let it be placed so that the angle *PTN* is not less than *T1.* Then, with the same construction as before, *PV =1= P'V'.*

And, since *T 4: L T1,*

.4N>:1?1'1,

and therefore NO <N/0, where P1N1 is the ordinate of Pi.

fence *PL <P1P2.*

But, *P1P2>P'F'.*

Therefore *PL>* 3P1',

so that *F,* the centre of gravity of the immersed portion of the solid, lies be­tween *P* and L.

Thus the solid will turn in the direction of diminishing the angle *PTN* until that angle becomes less than T1.

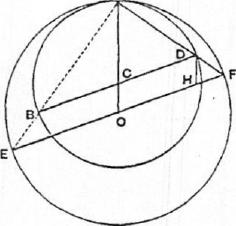
BOOK OF LEMMAS

**PROPOSITION 1**

*If two circles touch at A, and if BD, EF be parallel diameters in. them, A DF is* a *straight line.*

Let 0, C be the centres of the circles, and let *OC* be joined and produced to A. Draw *DH* par­allel to *AO* meeting *OF* in *II.*

**A**



Then, since *0II = CD = CA,*

and' *OF=0A,*

we have, by subtraction, *HF=CO=DH.*

Therefore Z */MP* = *Zing).*

Thus both' the triangles *CAD, HDF* are isos-

celes, and the third angles *ACD, DHF* in each

are equal. Therefore the equal angles in each

are equal to one another, and

*LADC= &DM.*

Add to each the angle *CDF,* and it follows that

*Z ADC+ ZCDF= LCDF-1- ZDFII*

*=* (two right angles).

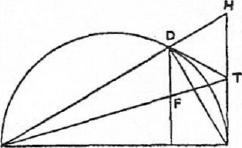
Hence *ADF* is a straight line.

The same proof applies if the circles touch externally.

**PROPOSITION 2**

*Let* ***AB*** *be the diameter of a semicircle, and let the tangents to it at B and at any other point D on it meet in T. If now DE be drawn perpendicular to AB, and if A7', DE meet in F,*

*DF =FE.*



**A**

a

Produce *AD* to meet *13T* produced in *H.* Then the angle *ADB* in the semi­circle is right; therefore the angle *BDH* is also right. And *TB, TD* are equal.

Therefore *'1'* is the centre of the semicircle on *BH* as diameter, which passes through D.

Hence *HT = TB.*

And, since *DE, HB* are parallel, it follows that *DF=FE.*

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PROPOSITION 3

*Let P be any point on a segment of a circle whose base is AB, and let PN be per-*

*pendicular to AR. Take D on AB so that AN =ND. If now PQ be an. arc equal*

*to the arc PA, and BQ be joined,*

*BQ, BD shall be equal.*

Join PA, *PQ, PD, DQ.*

Then, since the arcs *PA, PQ* are equal,

*PA = PQ.*

But, since *AN =ND,* and the angles at

iNT are right,

*PA =PD.*

Therefore *PQ = PD,*

and L *PQD = L PDQ.*

Now, since *A, P, Q, B* are coneyclic,

.L PAD- *Z PQB =* (two right angles),

whence L *1'DA + L PQB =* (two right angles)

= *ZPDA+ L PDB.*

Therefore L *PQB = L PDB;*

(Ind, since the parts, the angles *PQD, PDQ,* are equal,

*BQD= Z BDQ,*

and *BQ = BD.*

PROPOSITION 4

*If AB be the diameter of a semicircle and N any point on AR, and if semicircles be described within the first semicircle and haring AN, RN as diameters respec­tively, the figure included between the circumferences of the three semicircles is "what Archimedes called an apOnXos" ;1 and its area is equal to the circle on. PN as diameter, where PN is perpewlicida.r to AB and meets the original semicircle*

*in* P.

For

A *B2* = A N2 + *NI32+* 2A *N •NB*

*= A N2+N B2+* 2PA". But circles (or semicircles) are to one another as the squares of their radii (or diameters).

Hence

(semicircle on *AB) =* (sum of

semicircles on AN, *NB)*

+2(semicircle on *PN).*

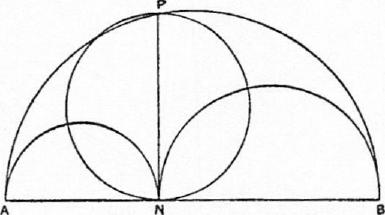
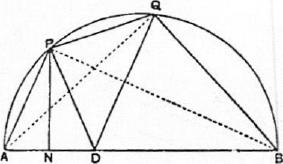
That is, the circle on *PN* as diameter is equal to the difference between the semicircle on *AB* and the sum of the semicircles on AN, NB, i.e. is equal to the area of the apfinXos.

PROPOSITION 5

*Let AB be the diameter of a semicircle, C any point on AB, and CD perpendicular toil, and let semicircles be described within the first semicircle and having AC, CB a's diameters. Then, if two circles be drawn touching CD on different sides and each touching two of the semicircles, the circles so drawn will be equal.*

Let one of the circles touch *CD* at *E,* the semicircle on *AB* in *F,* and the semicircle on *AC* in G.

lapsoo, is literally "a shoemaker's knife."



BOOK OI'' LEMMAS **161**

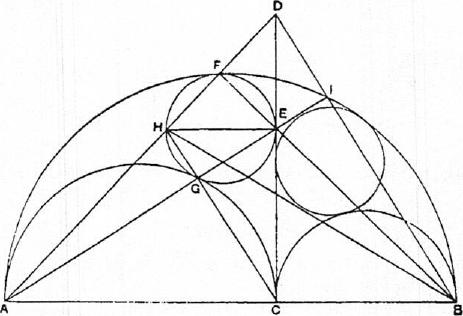
**I)ra**.**w** the diameter *Eli* of the circle, which will accordingly be perpendicular to *CD* and therefore parallel to *AB.*

Join *F11, HA,* and *FE, EB.* Then, by Prop. 1, *FHA, FEB* are both straight lines, since *EH, AB* are parallel.

For the same reason *AGE, CGH* are straight lines.

Let *Al,'* produced meet *CD* in *D,* and let *AE* produced meet the outer semi­circle in *I.* Join *131, ID.*

Then, since the angles *AFB,* A *CD* are right, the straight lines :l1), AB are such that the perpendiculars on each from the extremity of the other meet in the point *E.* Therefore, by the properties of triangles, *AE* is perpendicular to the line joining *B* to *D.*



But *AE* is perpendicular to *BI.*

Therefore *BID* is a straight line.

Now, since the angles at *G, I* are right, *CH* is parallel to *BD.*

Therefore *AB : BC = AD : DII*

*=AC :11E,*

so that *AC • CI3= AB • HE.*

In like manner, if *d* is the diameter of the other circle, we can prove that

*AC • CB= AB • d.*

Therefore *d= HE,* and the circles are equal.

PROPOSITION 6

*Let AB, the diameter of a semicircle, be divided at C so that AC =g-CB [or in any ratio]. Describe semicircles within the first semicircle and on AC, CB as diameters, and suppose* a *circle drawn touching all three semicircles. If GII be the diameter of this circle, to find the relation between Gil and AB.*

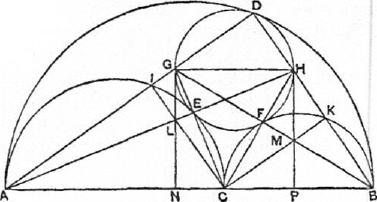
Let *Gil* be that diameter of the circle which is parallel to *AB,* and let the circle touch the semicircles on *AB, AC, CB* in *D, E, F* respectively.

Join *AG, GD* and *BH, HD.* Then, by Prop. 1*, AGD, BHD* are straight lines. For a like reason *AEII, BFG* are straight lines, as also are *CEG, CF II.*

Let AD meet the semicircle on *AC* in *I,* and let *BD* meet the semicircle on *CB* in *K.* Join *CI, CK* meeting *AE, BF* respectively in *L, Al,* and let *GL, JIM* produced meet. *AB in N, P* respectively.

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Now, in the triangle *AGC,* the perpendiculars from *A,* C on the opposite sides meet in *L.* Therefore, by the properties of triangles, *GLN* is perpendicular to *AC.*



Similarly *HMP* is perpendi­cular to *CB.*

Again, since the angles at *I, K,* D are right, *CK* is parallel to *AD,* and *CI* to *BD.*

Therefore *AC : CB = AL : LII =AN :NP,*

and *BC : CA = BM :MG =BP : PN.* Hence *AN :NP=NP :PB,* or *AN, NP, PB* are in continued proportion.

Now, in the case where AC *--gCB,*

*AN =IN P*

whence *BP : PN :NA : AB=4* :6:9:19.

Therefore *GH=NP=TegAB.*

And similarly *GH* can be found when *AC : CB* is equal to any other given

ratio.

PROPOSITION *7*

*If circles be circumscribed about and inscribed in a square, the circumscribed circle is double of the inscribed circle.*

For the ratio of the circumscribed to the inscribed circle is equal to that. of the square on the diagonal to the square itself, i.e. to the ratio 2 : 1.

PROPOSITION 8

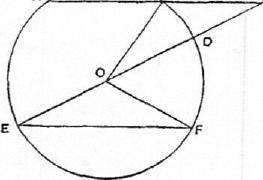
*If AB be any chord of a circle whose centre is 0, and if AB be produced to C so that*

*BC is equal to the radius; if further CO meet the circle in D and be produced to*

*meet the circle a second time in E, the arc AE will be equal to three times the arc*

*BD.*

Draw the chord *EF* parallel to *AB,* and



join *OB, OF.*

Then, since the angles *OEF, OFE* are

equal,

*LCOF =2 LOBE*

=2 L *BCO,* by parallels,

=2 *z .BOD,* Since *BC = BO.*

Therefore

*LBOF=*3 L *BOI),*

so that the arc *BF* is equal to three times the arc *BD.*

Hence the arc *A E,* which is equal to the arc *BF,* is equal to three times the

arc *BD.*

l'itorozirrioN 9

*If in a circle two chords AB, CD which do not pass through the centre intersect at right angles, then*

BOOK OF LEMMAS 163

*(arc AD)-1- (arc CB) = (arc AC)-F (arc DB).*

Let the chords intersect at *0,* and draw the diameter *EF'* parallel to *AB*

intersecting *CD* in *H. EF* will thus bisect *CD* at



right angles in *H,* and

(arc *ED)=* (arc *EC).*

Also *EDF, ECF* are semicircles, while

(arc *ED) =* (arc *EA) -F* (are AD).

Therefore

(sum of arcs *Cl", EA, AD)=* (arc of a semicircle).

And the arcs *AE, BF* are equal.

Therefore

(arc *CB)-1-* (arc *Al))=* (arc of a semicircle).

Hence the remainder of the circumference, the

sum of the arcs *AC, DB,* is also equal to a semi-

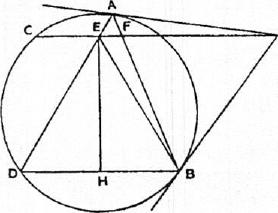
circle; and the proposition is proved.

l'aoposrrioN 10

*Suppose that* TA, *TB are two tangents to a circle, while TC cuts it. Let RD he the chord through B parallel to 71C, and let AD meet TC in E. Then, if EH be drawn perpendicular to BD, it will bisect it in H.*

Let *AB* meet *TC* in *F,* and join *RE.*

Now the angle *TAB* is equal to the angle in the alternate segment., i.e.



.L *TAB= LAUB*

*= L AET,* by parallels. Hence the triangles *EAT, AFT* have one angle equal and another (at *T)* com­mon. They are therefore similar, and *FT : A7' = A* : *ET.* Therefore *ET •TF =TA'*

*= TB2.*

It follows that the triangles *EBT, BFT* are similar.

Therefore *LTER= LITT*

*= Z TAB.*

But the angle *TEB* is equal to the angle *EI3D,* and the angle *TAB* was

proved equal to the angle *EDB.*

Therefore L *EDB = L E BD .*And the angles at *H* are right angles.

It follows that *BH =HD.*

PROPOSITION 11

*if two chords AB, CD in a circle intersect at right angles in a. point 0, not being*

*the centre, then*

*A024-B02-FCO2+ DO' =(diameter)2.*

Draw the diameter *CE,* and join *AC, CR, AD, BE.*

|  |  |
| --- | --- |
| 184 ARCIiIMEDES  Then the angle *CAO* is equal to the angle *CEB* in the same segment, and the angles *AOC, EBC* are right; therefore the triangles *AOC, EBC* are similar, and  *LACO= Z EC B*  It follows that the subtended arcs, and therefore the chords *AD, BE,* are equal.  Thus (A02+D02)+ *(B02-1- CO2) = AD2+BC2*  *= BE2+ BC' =CE2.* |  |

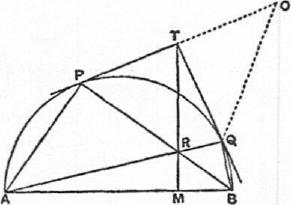
**PROPOSITION** 12

*If AB be the diameter of a semicircle, and TP, 7'Q the tangents to it from any*

*point T, and if AQ, BP be joined meeting in R, then TR is perpendicular to AB.*

Let *TI?* produced meet *AB* in *M,* and join *PA, QB.*

Since the angle *APB* is right,



*L PAB+ L PBA =* (a right angle)

*= LAQB.*

Add to each side the angle *RBQ,* and

*L PA B + LQBA=* (exterior) L *PRQ.*

But

*LTPR= LPAB,* and Z *TQR= ZQBA,*

in the alternate segments;

therefore L *TYR+ LTQR= ZPRQ.*

It follows from this that

TP= *TQ =TR.*

[For, if *PT* be produced to 0 so that *TO = TQ,* we have

*LTOQ= L TQO.*

And, by hypothesis, L *PRQ= L TPR+TQR.*

By addition, Z *POQ+ ZPRQ= L TPR+OQR.*

It follows that, in the quadrilateral *OPRQ,* the opposite angles are together equal to two right angles. Therefoie a circle will go round *OPQR,* and *T is* its centre, because *TP = TO =* 71Q. Therefore *TR=TP.]*

Thus Z *7'RP = L TPR= L PAM.*

Adding to each the angle *PRM ,*

*Z PAM + LPRH= T RP + LPRH*

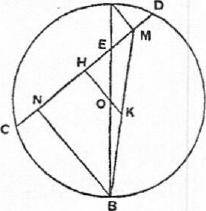
*=* (two right angles).

Therefore L *APR+ LAM!? =* (two right angles),

whence Z *A MR=* (a right angle).

**PROPOSITION** 13

**A**



*If a diameter AB of a circle meet any chord CD, not a diameter, in E, and if AM, BN be drawn perpendicu­lar to CD, then*

*CN =*

Let *0* be the centre of the circle, and *OH* perpen­dicular to *CD.* Join *BM,* and produce *HO* to meet *BM* in *K.*

Then *CH =IID.*

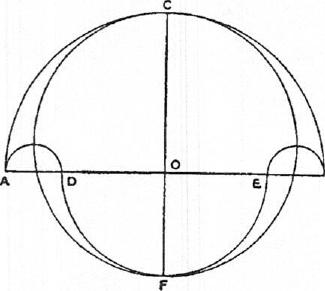
|  |  |
| --- | --- |
| And, by parallels, since  Therefore Accordingly | BOOK OF LEMMAS 165  *BO =OA, BK = KM. NH=HM. CN = DM.* |

PROPOSITION 14

*Let ACB be a semicircle on AB as diameter, and let AD, BE be equal lengths measured along AB from A, B respectively. On AD, BE as diameters describe semicircles on the side towards C, and on 1)E as diameter a semicircle on the op­posite side. Let the perpendicular to AB through 0, the centre of the first semi­circle, meet the opposite semicircles in C, F respectively.*

*Then shall the area of the figure bounded by the circumferences of all the semi­circles be equal to the area of the circle on CF as diameter.*

By Eucl. II. 10, since *ED* is bisected at *0* and produced to *A, EA2+AD2=2(E02+0A2),* and *CF=OA+OE=EA.* Therefore



*AB2-1-DE2=4(E02+0A2)=*

*2(CF2+AD2).*

But circles (and therefore semi­**s** circles) are to one another as the squares on their radii (or diameters). Therefore

(sum of semicircles on *AB, DE)*

= (circle on *CF) +* (sum of semicircles on *AD, BE).*

Therefore

(area of "salinon") = (area of circle on *CF* as diam.).

PROPOSITION 15

*Let AB be the diameter of a circle, AC a side of an inscribed regular pentagon, D*

*the middle point of the arc AC. Join CD and produce it to meet BA produced in*

*E; join AC, DB meeting in F, and draw FM perpendicular to AB. Then*

*EM = (radius of circle).*

Let 0 be the centre of the circle, and join *DA, DM, DO, CB.*

*Now L A BC =* (right angle),

and *Z ABD = Z DBC =* Wight angle),

whence L AOD=2.(right angle).

Further, the triangles *FCB, FMB* are equal in all respects.

Therefore, in the triangles *DCB, DMB,* the sides *CB, MB* being equal and

*BD* common, while the angles *CBI), MB!)* are equal,

*L BCD = L BALD = 2-* (right angle).

But *L BCD+ L BAD =* (two right angles)

*= ZBAD+ ZDAE  
= ZBMD+ Z DMA,*

so that L*DAE= Z BCD,*

and L*BAD= Z AMD.*

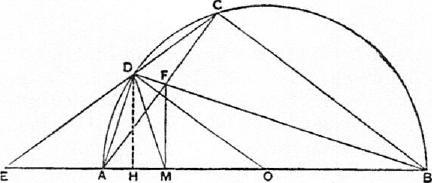
Therefore A D = *MD.*

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Now, in the triangle *DMO,*

*L MOD =* (right angle),

*Z DM0 =* \*(right angle).



Therefore L ODM = 1.(right angle) = *AOD;*

whence *OM = MD.*

Again L *EDA =* (supplement of *ADC)*

*= L C BA*

*=* (right angle)

=

Therefore, in the triangles *EDA, ODM,*

*L EDA = L ODM,*

*LEAD = L OM D,*

and the sides *AD, MD* are equal.

Hence the triangles are equal in all respects, and

*EA =MO.*

Therefore *EM =AO.*

Moreover *DE = DO;* and it follows that, since *DE* is equal to the side of an inscribed hexagon, and *DC* is the side of an inscribed decagon, *EC* is divided at *D* in extreme and mean ratio [i.e. *EC : ED = ED : DC];* "and this is proved in the book of the *Elements."* [Eucl. xtu. 9: "If the side of the hexagon and the side of the decagon inscribed in the same circle be put together, the whole straight line is divided in extreme and mean ratio, and the greater segment is the side of the hexagon."]

THE METHOD TREATING OF MECHANICAL  
PROBLEMS

"Archimedes to Eratosthenes greeting.

"I sent you on a former occasion some of the theorems discovered by me, merely writing out the enunciations and inviting you to discover the proofs, which at the moment I did not give. The enunciations of the theorems which I sent were as follows:

1. "If in a right prism with a parallelogrammic base a cylinder be inscribed which has its bases in the opposite parallelograms,' and its sides [i.e. four gen­erators] on the remaining planes (faces) of the prism, and if through the centre of the circle which is the base of the cylinder and (through) one side of the square in the plane opposite to it a plane be drawn, the plane so drawn will cut off from the cylinder a segment which is bounded by two planes and the sur­face of the cylinder, one of the two planes being the plane which has been drawn and the other the plane in which the base of the cylinder is, and the surface being that which is between the said planes; and the segment cut off from the cylinder is one sixth part of the whole prism.
2. "If in a cube a cylinder be inscribed which has its bases in the opposite parallelograms2 and touches with its surface the remaining four planes (faces), and if there also be inscribed in the same cube another cylinder which has its bases in other parallelograms and touches with its surface the remaining four planes (faces), then the figure bounded by the surfaces of the cylinders, which is within both cylinders, is two-thirds of the whole cube.

"Now these theorems differ in character from those communicated before; for we compared the figures then in question, conoids and spheroids and seg­ments of them, in respect to size, with figures of cones and cylinders: but none of those figures have yet been found to be equal to a solid figure bounded by planes; whereas each of the present figures bounded by two planes and surfaces of cylinders is found to be equal to one of the solid figures which are hounded by planes. The proofs then of these theorems I have written in this book and now send to you. Seeing moreover in you, as I say, an earnest student, a man of considerable eminence in philosophy, and an admirer [of mathematical in­quiry], I thought fit to write out foryou and explain in detail in the same book the peculiarity of a certain method, by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves; for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geom-

'The parallelogramS are apparently *squtzres.*2i.e. squares.

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etry afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge. This is a reason why, in the case of the theorems the proof of which Eudoxus was the first to discover, namely that the cone is a third part of the cylinder, and the pyramid of the prism, having the same base and equal height, we should give no small share of the credit to Democritus who was the first to make the assertion with regard to the said figure though he did not prove it. I am myself in the position of having first made the discovery of the theorem now to be published [by the method indicated], and I deem it necessary to expound the method partly be­cause I have already spoken of it and I do not want to be thought to have uttered vain words, but equally because I am persuaded that it will be of no little service to mathematics; for I apprehend that some, either of my contem­poraries or of my successors, will, by means of the method when once estab­lished, be able to discover other theorems in addition, which have not yet occurred to me.

"First then I will set out the very first theorem which became known to me by means of mechanics, namely that

*"Any segment of* a *section of* a *right-angled cone (i.e. a parabola) is four-thirds of the triangle which has the same base and equal height,*

and after this I will give each of the other theorems investigated by the same method. Then, at the end of the book, I will give the geometrical" [proofs of the propositions]...

[I premise the following propositions which I shall use in the course of the work.]

1. "If from [one magnitude another magnitude be subtracted which has not the same centre of gravity, the centre of gravity of the remainder is found by] producing [the straight line joining the centres of gravity of the whole magni­tude and of the subtracted part in the direction of the centre of gravity of the whole] and cutting off from it a length which has to the distance between the said centres of gravity the ratio which the weight of the subtracted magnitude has to the weight of the remainder." *[On the Equilibrium of Planes,* I. 8]
2. "If the centres of gravity of any number of magnitudes whatever be on the same straight line, the centre of gravity of the magnitude made up of all

of them will be on the same straight line." [Cf. *Ibid.* i. 5]

1. "The centre of gravity of any straight line is the point of bisection of the

straight line." [Cf. *Ibid.* I. 4]

1. "The centre of gravity of any triangle is the point in which the straight lines drawn from the angular points of the triangle to the middle points of the

(opposite) sides cut one another." *[Ibid.* I. 13, 14]

1. "The centre of gravity of any parallelogram is the point in which the

diagonals meet." *[Ibid.* I. 10]

1. "The centre of gravity of a circle is the point which is also the centre [of the circle]."
2. "The centre of gravity of any cylinder is the point of bisection of the axis."
3. "The centre of gravity of any cone is [the point which divides its axis so that] the portion [adjacent to the vertex is] triple [of the portion adjacent to the base]."

'1'IlE 'METHOD TREATING Oh MECHANICAL PROBLEMS 169

[All these propositions have already been] proved.' [Besides these I require also the following proposition, which is easily proved:

If in two series of magnitudes those of the first series are, in order, propor­tional to those of the second series and further], "the magnitudes [of the first series], either all or some of them, are in any ratio whatever [to those of a third series], and if the magnitudes of the second series are in the same ratio to the corresponding magnitudes [of a fourth series], then the sum of the magnitudes of the first series has to the sum of the selected magnitudes of the third series the same ratio which the sum of the magnitudes of the second series has to the sum of the (correspondingly) selected magnitudes of the fourth series." *[On Conoids and Spheroids,* Prop. 1.]

PROPOSITION 1

Let *ABC* be a segment of a parabola bounded by the straight line *AC* and

the parabola *ABC,* and let, *D* be the middle point of *AC.* Draw the straight line

*DBE* parallel to the axis of the parabola and join *AB, BC.*

Then shall the segment *ABC* be 4- of the triangle *ABC.*

From *A* draw *AKF* parallel to *DE,* and let the tangent to the parabola at *C*

meet *DBE* in *E* and *AKF* in *F.* Produce *CB* to meet *AF* in *K,* and again pro-

duce *CK* to *H,* making *KH* equal to *CK.*

Consider *CII* as the bar of a balance, *K* being its middle point.

Let *MO* be any straight line parallel to *ED,* and let it meet *CF, CK, AC* in

*M, N, O* and the curve in *P.*

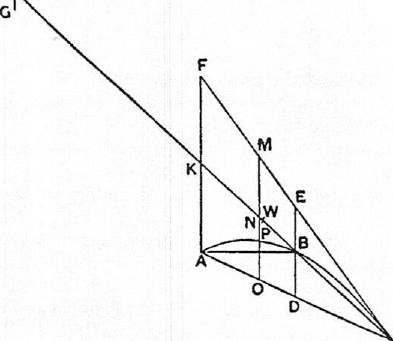
Now, since *CE* is a tangent. to the parabola and *CD* the semi-ordinate,

*EB=BD;*

"for this is proved in the Elements [of Conics]."2

H Since *FA, MO* are parallel to *ED,* it follows that

*FK=KA, MN=NO.*



Now, by the property of the parabola, "proved in a lem­ma,"

MO : *OP =CA : AO [Cf. Quad-rature of Parabola,* Prop. 5]

*=CK : KN*

[Encl. vi. 2] *=LIK : KN.*

Take a straight line *TG* equal to *OP,* and place it with its centre of gravity at *II,* so that *TH=I1G;* then, since *N* is the centre of gravity of tile straight line *MO,* and

MO : *TG=HK:KN,*

it follows that *TG* at *H* and *MO* at *N* will be in equilibrium about *K. [On the Equilibrium of Planes,* I. 6, 7]

'The problem of finding the centre of gravity of a cone is not solved in any extant work

of Archimedes.

21.e. the works on conies by Aristaeus and Euclid.

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Similarly, for all other straight lines parallel to *DE* and meeting the arc of the parabola, (1) the portion intercepted between *PC, AC* with its middle point on *KC* and (2) a length equal to the intercept between the curve and *AC* placed with its centre of gravity at *H* will be in equilibrium about *K.*

Therefore *K* is the centre of gravity of the whole system consisting (1) of all the straight lines as *MO* intercepted between *FC, AC* and placed as they actually are in the figure and (2) of all the straight lines placed at *H* equal to the straight lines as *PO* intercepted between the curve and *AC.*

And, since the triangle *CFA* is made up of all the parallel lines like *MO,* and the segment *CBA* is made up of all the straight lines like *PO* within the curve,

it follows that the triangle, placed where it is in the figure, is in equilibrium about *K* with the segment *CBA* placed with its centre of gravity at *H.* Divide *KC* at W so that *CK =* 3KW;

then TV is the centre of gravity of the triangle *ACP;* "for this is proved in the books an equilibrium" *(iv rocs I \_cropportKois).*

[Cf. On *the Equilibrium of Planes* r. 151

Therefore *PACT :* (segment *ABC) =7IK : KW*

=3 : 1.

Therefore segment *ABC = 3 CF.PA*

But. *AACF=4AABC.*

Therefore Segment *A BC = 4-PARC.*

"Now the fact here stated is not actually demonstrated by the argument

used; but that argument has given a sort of indication that the conclusion is true. Seeing then that the theorem is not demonstrated, but at the same time suspecting that the conclusion is true, we shall have recourse to the geometri­cal demonstration which I myself discovered and have already published."

PROPOSITION 2

We can investigate by the same method the propositions that.

1. *Any sphere is (in respect of solid content) four times the cone with base equal to a great circle of the sphere and height equal to its radius; and*
2. *the cylinder with base equal to a great circle of the sphere and height equal to the diameter is 11 times the sphere.*

*(1)* Let, *ABCD* he a great circle of a sphere, and *AC, BD* diameters at right angles to one another.

Let a circle be drawn about *BD* as diameter and in a plane perpendicular to A C,and on this circle as base let a cone he described with A as vertex. Let the surface of this cone he produced and then cut by a plane through *C* parallel to its base; the section will be a circle on *EP* as diameter. On this circle as base let a cylinder be erected with height and axis *AC,* and produce *CA* to *H,* making *AH* equal to *CA.*

Let *CH* be regarded as the bar of a balance, A being its middle point.

Draw any straight line *MN* in the plane of the circle *ABCD* and parallel to *BD.* Let *MN* meet the circle in *0, P,* the diameter *AC* in *8,* and the straight lines *AE, AP* in *Q, R* respectively. Join *AO.*

Through *MN* draw a plane at right angles to *AC;*

THE METHOD TREATING OF MECHANICAL PRO13LENIS **171**

this plane will cut the cylinder in a circle with diameter MN, the sphere in a

circle with diameter *OP,* and the cone in a circle with diameter *QR.*

Now, since *MS =AC,* and *QS= AS,*

*MS •SQ=CA -AS*

*—A02*

*=082+8Q2.*

And, since *HA = AC,*

*HA : AS=CA :AS*

*=MS : SQ*

=.ALS2 : *MS •SQ*

*=11S2 : (0S2+SQ2),*

from above,

= ItIN2 *(0P2+QR2)*

= (circle, diam. *MN) :* (circle, diam. *OP*

+circle, diam. *QR).*

That is,

*HA :AS =* (circle in cylinder) : (circle in sphere+circle in cone).

Therefore the circle in the cylinder, placed where it is, is in equilibrium, about A, with the circle in the sphere together with the circle in the cone, if both the latter circles are placed with their centres of gravity at *H.* Similarly for the three corre­sponding sections made by a plane perpendicular to *AC* and passing through any other straight line in the parallelogram *1,F* parallel to *EF.*

If we deal in the same way with all the sets of three circles in which

Mi **N** planes perpendicular to AC cut the

**OQ**

**SR P**

cylinder, the sphere and the cone,

**8**

**o**

and which make up those solids re­spectively, it follows that the cylin­der, in the place where it is, will be in equilibrium about *A* with

the sphere and the cone together, when both are placed with their centres of gravity at *II.*

Therefore, since *K* is the centre of gravity of the cylinder,

*HA : A K =* (cylinder) : (sphere + cone *A ER).*

But *I1A =* 2A *K,*

therefore cylinder = 2(sphere+ cone *A EF).*

Now cylinder = 3(cone *AEF);* [Euel. xII. **10]**

therefore cone *AEF = 2(sphere).*

But, since *EF = 2BD,*

cone *AEF=8(cone* A *BD);*

therefore sphere = 4(cone *ABD).*

(2) Through *B, D* draw *VBTV,* XDY parallel to *AC;*

and imagine a cylinder which has *AC* for axis and the circles on *VX,* WY as

diameters for bases.

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|  | **K,)** |

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**172** ARCIIINIEDES

Then cylinder V Y = 2(cylinder *VI))*

= 6(cone *ABU) [Euel. xii.* **10]  
=** (sphere), from above.

"From this theorem, to the effect that a sphere is four times as great as the cone with a great circle of the sphere as base and with height equal to the radius of the sphere, **I** conceived the notion that the surface of any sphere is four times as great as a great circle in it; for, judging from the fact that any circle is equal to a triangle with base equal to the circumference and height equal to the radius of the circle, I apprehended that, in like manner, any sphere is equal to a cone with base equal to the surface of the sphere and height equal to the radius."

PROPOSITION 3

By this method we can also investigate the theorem that

*A cylinder with base equal to the greatest circle in* a *spheroid and height equal*

*to the axis of the spheroid is 11 times the spheroid;*

and, when this is established, it is plain that

*If* any *spheroid be cut by a plane through the centre and at right angles to the*

*axis, the half of the spheroid is double of the cone which has the same base and the*

*same axis as the segment (i.e. the half of the spheroid).*

Let a plane through the axis of a spheroid cut its surface in the ellipse

*ABCD,* the diameters (i.e. axes) of which are *AC, BD;* and let *K* be the centre.

Draw a circle about *BD* as diame-

ter and in a plane perpendicular to

V **A X**

G

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| 6r |  |  |  |  | **D** |
| **/...1(** |  |  |  |  |  |

E w C

*AC;* imagine a cone with this circle as base and *A* as vertex produced and cut by a plane through *C* parallel to its base; the section will be a circle in a plane at right angles to *AC* and about *EF* as diameter.

Imagine a cylinder with the latter circle as base and axis *AC;* produce *CA* to *II,* making *AH* equal to *CA.*

Let *HC* be regarded as the bar of a balance, A being its middle point.

In the parallelogram *LF* draw any straight line MN parallel to *EF* meet­ing the ellipse in *0, P* and *AE, AF, AC* in *Q, R, S* respectively.

If now a plane be drawn through *MN* at right angles to *A C,* it will cut the cylinder in a circle with diame-

ter *MN,* the spheroid in a circle with diameter *OP,* and the cone in a circle with diameter *QR.*

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Since *HA= AC,*

*HA : AS=CA :AS*

*=EA : AQ*

*=MS : SQ.*

Therefore

*HA : AS =* MS' : *MS •SQ.*

But, by the property of the ellipse,

*AS •Se : 802=AK2:KB2*

*=AS2 :* SQ2;

therefore SQ2 : 802 = A *: AS • SC*

*=SQ2:SQ • QM,*

and accordingly *SO9= SQ • Q31.*

Add *SQ2* to each side, and we have

*802+8Q2=SQ •SM*

Therefore, from above, we have

HA : *AS= MS2 : (SO2+ SQ2)*

*(0P2+QR2)*

= (circle, diam. *MN) :* (circle, diam. OP-Fcircle, diam. QR).

That is,

*HA :.4S=* (circle in cylinder) : (circle in spheroid-I-circle in cone).

Therefore the circle in the cylinder, in the place where it is, is in equilibrium,

about A, with the circle in the spheroid and the circle in the cone together, if

both the latter circles are placed with their centres of gravity at *H.*

Similarly for the three corresponding sections made by a plane perpendicu-

lar to AC and passing through any of her straight line in the parallelogram *LP*

parallel to *EF.*

If we deal in the same way with all the sets of three circles in which planes

perpendicular to *AC* cut the cylinder, the spheroid and the cone, and which

make up those figures respectively, it follows that the cylinder, in the place

where it is, will be in equilibrium about *A* with the spheroid and the cone

together, when both are placed with their centres of gravity at *II.*

Therefore, since *K* is the centre of gravity of the cylinder,

*HA : AK=* (cylinder) : (spheroid+cone A *EF).*

But. *HA =2AK;*

therefore cylinder = 2(spheroid+cone *AEF).*

111(1 cylinder =3(cone *A EP);* [Eucl. xii. 101

therefore cone *AEI,' = 2(spheroid).*

But, since *EP =2BD,*

cone *AEF =* 8(cone *ADD);*

therefore spheroid = 4(cone *ABD),*

and half the spheroid = 2(cone *ABD).*

Through *B, D* draw *1713117, XDY* parallel to *AC;*

and imagine a cylinder which has *AC* for axis and the circles on *VX,* WY as

diameters for bases.

Then cylinder VY=2(cylinder VD)

=6(cone *ADD)*

=Rspheroid), from above. Q.E.D.

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Prtorosrmos 4

Any *segment of a right-angled conoid* (i.e. a *paraboloid of revolution) cut off by a*

*plane at right angles to the axis is 12 times the cone which has the same base and*

*the same axis as the segment.*

This can be investigated by our method, as follows.

I,et a paraboloid of revolution be cut by a plane through the axis in the

parabola *BA C;*

and let it also be cut by another plane at right angles to the axis and intersect-

ing the former plane in *BC.* Produce *DA,* the axis of the segment, to *II,* making

*HA* equal to *AD.*

Imagine that *HD* is the bar of a balance, A being its middle point.

The base of the segment being the circle on *BC* as diameter and in a plane

perpendicular to AD,

imagine (1) a cone drawn with the latter

circle as base and A as vertex, and (2) a

cylinder with the same circle as base and

A D as axis.

In the parallelogram *EC* let any straight

line *MN* be drawn parallel to *BC,* and

through MN let a plane be drawn at right

angles to *AD;* this plane will cut the cylin-

der in a circle with diameter *MN* and the

paraboloid in a circle with diameter *OP.*

Now, *BAC* being a parabola and *BD,*

*OS* ordinates,

*DA : AS= BD : 082,*

or *HA* : AS= MS2 **: AS02.**

Therefore

*:* AS= (circle, rad. *MS) :* (circle, rad. *OS)*

= (circle in cylinder) : (circle in paraboloid).

Therefore the circle in the cylinder, in the place where it is, will be in equi­librium about A with the circle in the paraboloid, if the latter is placed with its centre of gravity at *H.*

Similarly for the two corresponding circular sections made by a plane per­pendicular to AD and passing through any other straight line in the parallelo­gram which is parallel to *BC.*

Therefore, as usual, if we take all the circles making up the whole cylinder and the whole segment and treat them in the same way, we find that the cylinder, in the place where it is, is in equilibrium about A with the segment placed with its centre of gravity at II.

If *K* is the middle point of AD, *K* is the centre of gravity of the cylinder;

therefore *HA : AK=* (cylinder) : (segment).

Therefore cylinder = 2(segment).

And cylinder = 3(cone *ABC);* [Encl. **XII.** 10]

therefore segment = (cone *A BC).*

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PuoPosrrioN 5

*The centre of gravity of a segment of a right-angled conoid (i.e. a paraboloid of*

*revolution) cut off* by *a plane at right angles to the axis is on the straight line which*

*is the axis of the segment, and divides the said straight line in such a way that the*

*portion of it adjacent to the vertex is double of the remaining portion.*

This can be investigated by the method, as follows.

Let a paraboloid of revolution be cut by a plane through the axis in the

parabola *BAC;*

and let it also be cut by another. plane at right angles to the axis and intersect-

ing the former plane in *13C.*

Produce *DA,* the axis of the segment, to *H,* making *HA* equal to *AD;* and

imagine *1)11* to be the bar of a balance, its middle. point being A.

The base of the segment being the circle on *13C* as diameter and in a plane

perpendicular to .41),

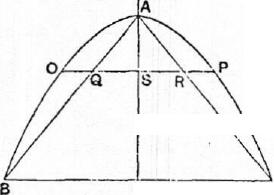
imagine a cone with this circle as base and A as vertex, so that *AB, AC* are

generators of the conc.

In the parabola let any double ordinate

H *OP* be drawn meeting .4/3, A *D, AC in Q, 8,*

*1?* respectively.



Now, from the property of the parabola,

BD2 : *OS'* = DA : *AS*

*= 13D : QS*

*=131)= : BD •* QS.

Therefore OS2= *BD • QS,*

*BD : OS=08 : QS,*

whence R D : QS = 0S2 : QS'.

But *ED : QS =* .4 D ::1*5*

*=1111 :AS.*

Therefore *HA :AS=OS' :* QS'

K = *OP' :*

If now through *OP* a plane he drawn at right angles to AD, this plane cuts the par-c aboloid in a circle with diameter *OP* and

the cone in a circle with diameter Qll.

We see therefore that. *HA :AS=* (circle, diam. *OP) :* (circle, diam. QH)

= (circle in paraboloid) : (circle in cone); and the circle in the paraboloid, in the place where it is, is in equilibrium about :1 with the circle in the cone placed with its centre of gravity at *H.*

Similarly for the two corresponding circular sections made by a plane per­pendicular to AD and passing through any other ordinate of the parabola.

Dealing therefore in the same way with all the circular sections which make up the whole of the segment of the paraboloid and the cone respectively, we see that the segment of the paraboloid, in the place where it is, is in equilibrium about *A.* with the cone placed with its centre of gravity at *II.*

Now, since II is the centre of gravity of the whole system as placed, and the centre of gravity of part of it , namely the cone, as placed, is at *H,* the centre of gravity of the rest, namely the segment, is at a point *K* on HA produced

such that *11A : AK =* (segment) : (cone).

But segment = (cone). [Prop. **4]**

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Therefore IIA = xA*K;*that is, *K* divides *AD* in such a way that AK *=2KD.*

PaorosrrioN 6

*The centre of gravity of any hemisphere [is on the straight line which] is its axis,*

*and divides the said straight line in such* a *way that the portion of it adjacent to the*

*surface of the hemisphere has to the remaining portion the ratio which* 5 *has to* 3.

Let a sphere be cut by a plane through its centre in the circle *4WD;*

let *AC, BD* he perpendicular diameters of this circle,

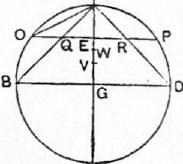
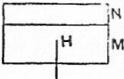
and through *BD* let a plane be drawn at right angles to A *C.*

The latter plane will cut the sphere in a circle on *BD* as diameter.

Imagine a cone with the latter circle as base and A as vertex.

Produce *CA* to *II,* making A *H* equal to *CA,* and let *//C* be regarded as the

bar of a balance, A being its middle point.



In the semicircle BAD, let any straight line *OP* be

drawn parallel to *BD* and cutting *AC* in *E* and the two

generators. *A B, Al)* of the cone in *Q, R* respectively.

Join A*O.*

Through *OP* let. a plane be drawn at right angles to

A *C;*

this plane \Viii cut the hemisphere in a circle with di-

ameter *OP* and the cone in a circle with diameter *QR.*

*Now*

*IIA :AE=AC:AE*

*=AO' : AE2*

*= (0E2+:1E2) : AE2*

*(0E2+QE2) : QE2*

= (circle, dizun. OP circle, diam. : (circle, diam. Therefore the circles with diameters *OP, QR,* in the

places where they are, are in equilibrium about A with the circle with diameter Ql? if the latter is placed with its centre of gravity at. *II.*

And, since the centre of gravity of the two circles with diameters *VP, QR* taken together, in the place where they are, is . .

[There is a lacuna here; but the proof can easily be completed on the lines of the corresponding but more difficult case in Prop. 8.

We proceed thus from the point where the circles with diameters *OP, QR,* in the place where they are, balance, about *A,* the circle with diameter *QR* placed with its centre of gravity at *H.*

A similar relation holds for all the other sets of circular sections made by other planes passing through points on AG and at right angles to *AG.*

Taking then all the circles which fill up the hemisphere *BAD* and the cone *ABD* respectively, we find that

the hemisphere *BAD* and the cone *ABD,* in the places where they are, together balance, about A, a cone equal to *ABD* placed with its centre of gravity at *H.* Let the cylinder *111+N* be equal to the cone *ABD.*

Then, since the cylinder *M+N* placed with its centre of gravity at *H* bal­ances the hemisphere *BAD* and the cone *ABD* in the places where they are, suppose that the portion M of the cylinder, placed with its centre of gravity at

TIIE METHOD TREATING OI'' MECHANICAL PROBLEMS *177*

*H,* balances the cone A *131)* (alone) in the place where it, is; therefore the por­tion N of the cylinder placed with its centre of gravity at *II* balances the hemisphere (alone) in the place where it is.

Now the centre of gravity of the cone is at a point *V* such that *AG =4GV,* therefore, since *M* at *II* is in equilibrium with the cone,

*ill :* (cone) = 4AG *: HA = C : AC,*

whence M = (cone).

But 1/-1-N.= (cone); therefore N= „(cone).

Now let the centre of gravity of the hemisphere be at. TV, which is somewhere

on AG.

Then, since *Nat II* balances the hemisphere alone,

(hemisphere) : N= *HA : ATV.*

But the hemisphere *BAD=* twice the cone *ABD;*

*[On the Sphere and Cylinder* I. 34 and Prop. 2 above]

and N = 4(cone), from above.

Therefore 2 : s *= HA : A TV*

*=2AG : ATV,*

whence *AlV =r1,-AG,* so that TV divides *AG* in such a way that

: !VG= 5 : 3.]

PROPOSITION 7

We can also investigate by the same method the theorem that

[Any *segment of a sphere has] to the cone [with the same base and height the ratio which the sum of the radius of the sphere and the height of the complementary segment has to the height of the complementary segment.]*

[There is a lacuna here; but all that is missing is the construction, and the construction is easily understood by means of the figure. *BAD is* of course the

H segment of the sphere the vol-

ume of which is to be compared

with the volume of a cone with the same base and height.]

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The plane drawn through *MN* and at right angles to *AC* will cut the cylinder in a circle with diameter *MN,* the seg­ment of the sphere in a circle with diameter *OP,* and the

N cone on the base *EIS'* in a circle  
with diameter QR.

In the same way as before [cf. Prop. 2] we can prove that the circle with diameter MN, in the place where it is, is in equilibrium about *A* with the two circles with diameters *01', QR* if these circles are both moved and placed with their centres of gravity at *II.*

The same thing can be proved of all sets of three circles in which the cylin-

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der, the segment of the sphere, and the cone with the common height AG are all cut by any plane perpendicular to A *C.*

Since then the sets of circles make up the whole cylinder, the whole segment of the sphere and the whole cone respectively, it follows that the cylinder, in the place where it is, is in equilibrium about A with the sum of the segment of the sphere and the cone if both are placed with their centres of gravity at *H.*

Divide *AG* at TV, *V* in such a way that

A W= *1VG, AV =* 3 *VG.*

Therefore *TV* will he the centre of gravity of the cylinder, and V will be the centre of gravity of the cone.

Since, now, the bodies are in equilibrium as described,

(cylinder) : (cone A.EF-Fsegment *BAD* of sphere) *=11A :* A W.

[The rest of the proof is lost.; but it can easily be supplied thus: We have

(cone AEF-Fsegmt. *BAD) :* (cylinder) =A W *:AC*

*=ATV •AC : AC2.*

But (cylinder) : (cone *AEF) = AC2 : IEG2*

*=AC2 :AAG2.*

Therefore, *ex aequali,*

(cone *A* EP-1-segmt. *BAD) :* (cone *A EF) = ATV •AC : 3AG2*

*=1AC :*

whence (segmt. *BAD) :* (cone *A EF)=* (1AC-3sAG) :

Again (cone *A EP) :* (cone *A BD) = EG2 : DG2*

*=AG' :AEG • GC =AG :G*

*=1AG:*?GC.

Therefore, *ex aequali,*

(segment *BAD) :* (cone ABD)=(1AC *—1AG) :1GC*

=RAC—AG) : *GC*

*=(1.4C+GC) :GC.* Q.E.D.1

PROPOSITION 8

[The enunciation, the setting-out, and a few words of the construction are missing.

The enunciation however can be supplied from that of Prop. 9, with which it must be identical except that it cannot refer to "any segment," and the presumption therefore is that the proposition was enunciated with reference to one kind of segment only, i.e. either a segment greater than a hemisphere or a segment less than a hemisphere.

Heiberg's figure corresponds to the case of a segment greater than a hemi­sphere. The segment investigated is of course the segment *BAD.* The setting-out and construction are self-evident from the figure.]

Produce A*C* to *II, 0,* making *HA* equal to *AC* and *CO* equal to the radius of the sphere;

and let *HC* be regarded as the bar of a balance, the middle point being *A.*

In the plane cutting off the segment describe a circle with *G* as centre and radius *(GE)* equal to *AG;* and on this circle as base, and with *A* as vertex, let a cone be described. *AE, A F* are generators of this cone.

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Draw *Kb,* through any point Q on *AG,* parallel to *EF* and cutting the seg-

ment in *K, I.,* and *A E, A* in *I?, P* respectively. Join *AK.*

Now

/1.1 *:AQ=CA : AQ*

*= AK2 AQ2*

= (A-Q2 4\_ Q..1 2) QA 2

= (K(22-1-PQ2) : *PQ2*

= (circle, dia.m. diam. *PR) :* (circle, diam. *PR).*

Imagine a circle equal to the circle with diam­eter *PIt* placed with its centre of gravity at *II;* herefore the circles on diameters *Kb, PI?,* in the places where they are, are in equilibrium about A with the circle with diameter *PI?* placed with its centre of gravity at *II.*



Similarly for the corresponding circular sec­tions made by any other plane perpendicular to *AG.*

Therefore, taking all the circular sections which make up the segment *ABD* of the sphere and the cone *AEI?* respectively, we find that the segment *ADD* of the sphere and the cone *A EF,* in the places where they are, are in equi­librium with the cone *AEF* assumed to be placed with its centre of gravity at *H.*

Let the cylinder *411+N* be equal to the cone AU which has A for vertex and the circle on *El"* as diameter for base.

Divide *AG* at V so that

.4G=4VG;

therefore *V* is the centre of gravity of the cone *ARP;* "for this has been proved before."

Let the cylinder -}-N be cut by a plane perpendicular to the axis in such a way that the cylinder A/ (alone), placed with its cent re of gravity at *II,* is in equilibrium with the cone *AEF.*

Since M-FN suspended at *H* is in equilibrium with the segment. Al BD of the sphere and the cone *AEF* in the places where they are,

while *M,* also at *II,* is in equilibrium with the cone *AEF* in the place where it is, it follows that

N at *H* is in equilibrium with the segment. A *BD* of the sphere in the place where it is.

Now (segment *A lID* of sphere) : (cone A *BD)=* 0(7 : GC;

"for this is already proved" [Cf. *On the Sphere and Cylinder* Ir. 2 (','or. as well

as Prop. 7 *ante].*

And (cone *ADD) :* (cone *A El")*

= (circle, diam. *BD) :* (circle, diam. *EP) =BD' :EP'*

*= :(;E'  
=CG •* GA : GA'

*=CG* : G.4.

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Therefore, *cx acquali,*

(segment *ABD* of sphere) : (cone .4 *EF)=OG :GA.*

Take a point TV on *AG* such that

*ATV :TVG= (GA+4GC) : (GA+2GC).*

We have then, inversely,

*GW :TVA=(2GC-FGA):(4GC+GA),*

and, *componendo,*

*GA : ATV =(6GC-F2GA) : (4GC ±* GA).

But GO= *-}(6GC+2GA),* [for *GO—GC =1(CG+GA)]*

and *CV* = (-1GC-FGA);

therefore *GA :* A tV=OG : CV,

and, alternately and inversely,

*OG* : GA = *CV* : TVA .

It follows, from above, that

(segment *ABD* of sphere) : (cone *AEF)=CV :WA.*

Now, since the cylinder 111 with its centre of gravity at *11* is in equilibrium

about *A* with the cone *AEF* with its centre of gravity at V,

(cone *AEP) :* (cylinder M) =*11A : A V*

*=CA :* AV;

and, since the cone *AEF =* the cylinder we have, *dividendo* and *inver-*

*tendo,* (cylinder *M):* (cylinder *N)= AV : CV.*

Hence, *componendo,*

(cone *A EF) :* (cylinder N) =CA *: C*

*=HA : CV.*

But it was proved that

(segment *ABD* of sphere) : (cone A *EF)=CV :WA;*

therefore, *cx acquali,*

(segment *ABD* of sphere) : (cylinder N)=.H.4 : *A* IV.

And it was above proved that the cylinder N at *H is* in equilibrium about A

with the segment *ABD,* in the place where it is;

therefore, since *H* is the centre of gravity of the cylinder *N, TV* is the centre

of gravity of the segment *ADD* of the sphere.

PROPOSITION 9

In the same way we can investigate the theorem that

*The centre of gravity of any segment of a sphere is on the straight line which is the axis of the segment, and divides this straight line in such a way that the part of it adjacent to the vertex of the segment has to the remaining part the ratio which the sum of the axis of the segment and four times the axis of the complementary segment has to the sum of the axis of the segment and double the axis of the com­plementary segment.*

[As this theorem relates to *"any* segment" but states the same result as that proved in the preceding proposition, it follows that Prop. 8 must have related to one kind of segment, either a segment greater than a semicircle (as in Hei-berg's figure of Prop. 8) or a segment less than a semicircle; and the present proposition completed the proof for both kinds of segments. It would only require a slight change in the figure, in any case.]

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PROPOSITION 10

By this method too we can investigate the theorem that

*[A segment of an obtuse-angled conoid (i.e.* a *hyperboloid of revolution) has to the cone which has] the same base [as the segment and equal height the same ratio as the sum of the axis of the segment and three times] the "annex to the* axis" (i.e. *half the transrerse axis of the hyperbolic section through the axis of the hyper-boloid, or, in other words, the distance between the vertex of the segment and the vertex of the enveloping cone) has to the sum of the axis of the segment and double of the "annex"* [this is the theorem proved in *On Conoids and Spheroids,* Prop. 23], "and also many other theorems, which, as the method has been made clear by means of the foregoing examples, I will omit, in order that I may now pro­ceed **to** compass the proofs of the theorems mentioned above."

PROPOSITION **1***1*

*If in a right prism with square bases a cylinder be inscribed having its bases in opposite square faces and touching with its surface the remaining four parallelo-grammic faces, and if through the centre of the circle which is the base of the cylin­der and one side of the opposite* square *face* a *plane* be *drawn, the* figure cut off by *the plane so drawn is one sixth part of the whole prism.*

"This can be investigated by the method, and, when it is set out, **I** will go back to the proof of it by geometrical considerations."

[The investigation by the mechanical method is contained in the two Propo­sitions, 11, 12. Prop. 13 gives another solution which, although it contains no mechanics, is still of the character which Archimedes regards as inconclusive, since **it** assumes that the solid is actually made up of parallel plane sections and that an auxiliary parabola is actually *made* up of parallel straight lines in it. Prop. 14 added the conclusive geometrical proof.]

Let there be a right prism with a cylinder in­scribed as stated.



Let the prism be cut through the axis of the

**E** prism and cylinder by a plane perpendicular to

the plane which cuts off the portion of the cylin­der; let this plane make, as section, the parallelo­gram *a 13,* and let it cut the plane cutting off the portion of the cylinder (which plane is perpen­dicular to *A13)* in the straight line *BC.*



Let *CD* be the axis of the prism and cylinder,

**A V Y** let *EF* bisect it at right angles, and through *EF*let a plane be drawn at right angles to *CD;* this plane will cut the prism in a square and the cylinder in a circle.

Let *MN* be the square and *OPQ1* the circle, and let the circle touch the sides of the square in 0, ***P,*** Q, II *[F, E* in the first figure are identical with 0, Q respectively]. Let *II* be the centre of the circle.

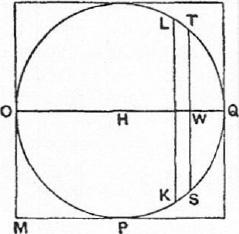
Let *KL* be the intersection of the plane through *EP'* perpendicular to the axis of the cylinder and the plane cutting off the portion of the cylinder; *KL* is bisected by *OHQ* [and passes through the middle point of 1/(2].

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Let any chord of the circle, as *ST,* be drawn perpendicular to *HQ,* meeting *HQ* in W;

and through *ST* let a plane be drawn at right angles to *OQ* and produced on both sides of the plane of the circle *OPQR.*

The plane so drawn will cut the half cylinder having the semicircle *PQR* for section and the axis of the prism for height in a parallelogram, one side of which is equal to *ST* and another is a generator of the cylinder; and it will also cut the portion of the cylinder cut off in a parallelogram, one side of which is equal to *ST* and the other is equal and parallel to *UV* (in the first figure).



UV will be parallel to BY and will cut off, along *Elf* in the parallelogram *DE,* the seg­ment *EI* equal to Q1V.

Now, since *EC* is a parallelogram, and *VI* is parallel to *GC,*

*EG :GI =YC :CV -*

*=BY : V*

*= (0* in half cyl.) : *(0* in portion of cyl.).

And EG *=HQ, GI= QH =OH;*

therefore *011:11* IV = *(0* in half cyl.) : *(O* in portion).

Imagine that the parallelogram in the portion of the cylinder is moved and placed at 0 so that 0 is the centre of gravity, and that OQ is the bar of a balance, *II* being its middle point.

Then, since IV is the centre of gravity of the parallelogram in the half cyl­inder, it follows from the above that the parallelogram in the half cylinder, in the place where it is, with its centre of gravity at IV, is in equilibrium about *H* with the parallelogram in the portion of the cylinder when placed with its cent re of gravity at *0.*

Similarly for the other parallelogrammic sections made by any plane per­pendicular to *OQ* and passing through any other chord in the semicircle *PQR* perpendicular to *OQ.*

1f then we take all the parallelograms making up the half cylinder and the portion of the cylinder respectively, it follows that the half cylinder, in the place where it is, is in equilibrium about *H* with the portion of the cylinder cut off when the latter is placed with its centre of gravity at 0.

PROPOSITION 12

Let the parallelogram (square) .11 11% perpendicular to the axis, with the circle *OPQR* and its diameters *OQ, PR,* be drawn separately.

Join *IIG, UM,* and through them draw planes at right angles to the plane of the circle, producing them on both sides of that plane.

This produces a prism with triangular section *CHM* and height equal to the axis of the cylinder; this prism is I of the original prism circumscribing the cylinder.

Let *LK, UT* be drawn parallel to *OQ* and equidistant from it, cutting the circle in *K, T, RP in S, F,* and *CH, HM* in IV, *V* respectively.

Through *LK, UT* draw planes at right angles to *PR,* producing them on both sides of the plane of the circle;

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these planes produce as sections in the half cylinder *PQR* and in the prism *GHM* four parallelograms in which the heights are equal to the axis of the

cylinder, and the other sides are equal to *KS, TF, LW, UV* respectively

[The rest of the proof is missing, but, as Zeuthen says, the result obtained and the method of arriving at it are plainly indicated by the above.

Archimedes wishes to prove that the half cylinder *PQR,* in the place where it is, balances the prism *GH.41,* in the place where it is, about *II* as fixed point.

He has first to prove that the elements (1) the parallelogram with side = *KS* and (2) the parallelogram with side = *L1V,* in the places where they are, balance about 5, or, in other words that the straight lines *SK, LW,* in the places where they are, balance about *S.*

(radius of circle *OPQR)2=sK2+ 3112,*

SL2 = SK2 S If".

**LiS2 AS W2 =***S10,*

G R N

|  |  |
| --- | --- |
| L  0 |  |

P

Now

or

Therefore

and accordingly *(LS-I-SW) •LW =SK2,*

whence 1-(LS+SIV) *:1,SK=SK : LW.*

And ..-(LS-FSIV) is the distance of the centre of gravity of LW from 8, while *ISK* is the distance of the centre of gravity of *8K* from *S.*

Therefore *SK* and *L W,* in the places where they are, balance about S. Similarly for the corresponding parallelograms.

Taking *all* the parallelogrammic elements in the half cylinder and prism respectively, we find that

the half cylinder *PQR* and the prism *GH.111,* in the places where they are re­spectively, balance about *II.*

From this result and that of Prop. 11 we can at once deduce the volume of the portion cut off from the cylinder. For in Prop. 11 the portion of the cylin­der, placed with its centre of gravity at *0,* is shown to balance (about *H)* the half-cylinder in the place where it is. By Prop. 12 we may substitute for the half-cylinder in the place where it is the prism *GHM* of that proposition turned the opposite way relatively to *RP.* The centre of gravity of the prism as thus placed is at a point (say *Z)* on *HQ* such that. *HZ=* 3HQ.

Therefore, assuming the prism to be applied at its centre of gravity, we have

(portion of cylinder) : (prism) = *IHQ :OH*

**=2 :3;**

therefore (portion of cylinder) = i(prism GHM)

= \*(original prism).

PROPOSITION 13

Let there he a right prism with square bases, one of which is *A BCD;* in the prism let a cylinder be inscribed, the base of which is the circle *EFGH*

touching the sides of the square *ABCD* in *E, G, H.*

Through the centre and through the side corresponding to *CD* in the square

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face *opposite* to *ABCD* let a plane be drawn; this will cut, off a prism equal

to of the original prism and formed by three parallelograms and two triangles,  
the triangles forming opposite faces.

In the semicircle *ERG* describe the parabola which has *FK* for axis and passes through *E, G;* draw *MN* parallel to *KI?* meeting *GE* in M, the para­bola in L, the semicircle in *0* and *CD* in *N.*

Then *MN • NL = NIA~2;*

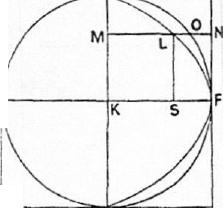
"for this is clear." [Cf. Apollonius, *Conics* I. 11]  
[The parameter is of course equal to *GK* or *KF.]*

Therefore MN : *NL=GK2 : LS2.*

Through *MN* draw a plane at right angles to

*EG ;*

this will produce as sections (1) in the prism cut off from the whole prism a right-angled triangle, the base of which is *MN,* while the perpendicular is perpendicular at *N* to the plane *ABCD* and



equal to the axis of the cylinder, and the hypot- **H** enuse is in the plane cutting the cylinder, and (2) in the portion of the cylinder cut off a right-angled triangle the base of which is *MO,* while the perpendicular is the generator of the cylinder

perpendicular at *0* to the plane *KN,* and the **A**

**0**

hypotenuse is

[There is a lacuna here, to be supplied as follows.

Since *MN : NL=GK2 :1,82*

*= MN2 : LS',*

*it* follows that *MN ; ML=MN2 : (MN2 — LS2)*

*= MN2 : (M N2— MK')*

*= : MO'.*

But the **triangle** (1) in the prism is to the triangle (2) in the portion of the

cylinder in the ratio of */1/N2* : *M02.*

Therefore (A in prism) : (A in portion of cylinder)

*=MN : ML*

= (straight line in rect. *DG) :* (straight line in parabola).

We now take all the corresponding elements in the prism, the portion of the

cylinder, the rectangle *DO* and the parabola *ERG* respectively];

and it will follow that

(all the As in prism) : (all the As in portion of cylinder)

= (all the str. lines in *ODG) :* (all the straight lines between parabola and *EG).*

But the prism is made up of the triangles in the prism, [the portion of the

cylinder is made up of the triangles in it], the parallelogram *DC* of the straight

lines in it parallel to *KF,* and the parabolic segment of the straight lines paral-

lel to *KF* intercepted between its circumference and *EG;*

therefore (prism) : (portion of cylinder)

*=(C3GD) :* (parabolic segment *ERG).*

But *DOD =* -(parabolic segment *ERG);*

"for this is proved in my earlier treatise." *[Quadrature of Parabola]*

Therefore prism = 4(portion of cylinder).

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If then we denote the portion of the cylinder by 2, the prism is 3, and the original prism circumscribing the cylinder is 12 (being 4 times the other prism); therefore the portion of the cylinder= i(original prism). **Q.E.D.** [The above proposition and the next are peculiarly interesting for the fact that the parabola is an auxiliary curve introduced for the sole purpose of analytically reducing the required eubature to the known quadrature of the parabola.]

PnoposrrioN 14

Let there be a right prism with square bases [and a cylinder inscribed therein

having its base in the square *ABCD* and touching its sides at *E, G, 11;*let the cylinder be cut by a plane through *EG* and the side corresponding to *CD* in the square face opposite to *ABCD1*

This plane cuts off from the prism a prism, and from the cylinder a portion of it.

It can be proved that the portion of the cylinder cut off by the plane is \* of the whole prism.

But we will first prove that it is possible to inscribe in the portion cut off from the cylinder, and to circumscribe about it, solid figures made up of prisms which have equal height and similar triangular bases, in such a way that the circumscribed figure exceeds the inscribed by less than any assigned magnitude.. ...

But it was proved that

(prism cut off by oblique plane) <1(figure inscribed in portion of cylinder).

Now (prism cut off) : (inscribed figure)

*=ODG : (Os* inscribed in parabolic segment);

therefore *ODG<1(0s* in parabolic segment):

which is impossible, since "it has been proved elsewhere" that the parallelo-

gram *DG* is 1- of the parabolic segment.

Consequently

not greater.

And (all the prisms in prism cut off)

: (all prisms in circumscr. figure)

= (all *Os* in *ODG) :* (all *Os* in fig. circumscr. about parabolic segmt.);

therefore

(prism cut off) : (figure circumscr. about portion of cylinder)

*= (ODG) :* (figure circumscr. about parabolic segment).

But the prism cut off *by* the oblique plane is> a of the solid figure circum-

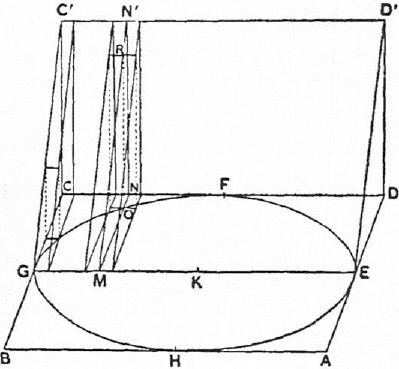
scribed about the portion of the cylinder... :.

[There are large gaps in the exposition of this geometrical proof, but the way in which the method of exhaustion was applied, and the parallelism be­tween this and other applications of it, are clear. The first fragment shows that solid figures made up of prisms were circumscribed and inscribed to the por­tion of the cylinder. The parallel triangular faces of these prisms were perpen­dicular to *GE* in the figure of Prop. 13; they divided *GE* into equal portions of the requisite smallness; each section of the portion of the cylinder by such a

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plane was a triangular face common to an inscribed and a circumscribed right prism. The planes also produced prisms in the prism cut off by the same oblique plane as cuts off the portion of the cylinder and standing on *CD* as base.

The number of parts into which the parallel planes divided *GE* was made great enough to secure that the circumscribed figure exceeded the inscribed figure by less than a small assigned magnitude.



The second part of the proof began with the assumption that the portion of

the cylinder is > 3 of the prism cut off; and this was proved to be impossible,

by means of the use of the auxiliary parabola and the proportion

*MN : ML= MN2 : MO2*

which are employed in Prop. 13.

We may supply the missing proof as follows.

In the accompanying figure are represented (1) the first element-prism circumscribed to the portion of the cylinder, (2) two element-prisms adjacent to the ordinate OM, of which that on the left is circum­scribed and that on the right (equal to the other) inscribed, (3) the corresponding element-prisms forming part of the prism cut off *(CC'GEDD')* which is 4 of the original prism.

In the second figure are shown element-rectangles circumscribed and inscribed to the auxiliary parab­ola, which rectangles correspond exactly to the cir­cumscribed and inscribed element-prisms represented in the first figure (the length of *GM* is the same in both figures, and the breadths of the • element-rec­tangles are the same as the heights of the element-

prisms) ; the corresponding element-rectangles form- **E 0**ing part of the rectangle *CD* are similarly shown.

For convenience we suppose that *CE* is divided into an even number of equal parts, so that *GK* contains an integral number of these parts.

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For the sake of brevity we will call each of the two element-prisms of which 031 is an edge "el. prism (0)" and each of the element-prisms of which *MNN'* is a common face "el. prism (N)." Similarly we will use the corresponding abbreviations "el. rect. *(L)"* and "el. rect. *(N)"* for the corresponding elements in relation to the auxiliary parabola as shown in the second figure.

Now it is easy to see that the figure made up of all the inscribed prisms is less than the figure made up of the circumscribed prisms by twice the final circumscribed prism adjacent to *PK,* i.e. by twice "el. prism (N)"; and, as the height of this prism may be made as small as we please by dividing *UK* into sufficiently small parts, it follows that inscribed and circumscribed solid figures made up of element-prisms can be drawn (Hireling by less than any assigned solid figure.

1. Suppose, if possible, that

(portion of cylinder) > 3 (prism cut off),

or (prism cut off) <Rportion of cylinder).

Let (prism cut off) =4(portion of cylinder — *X),* say.

Construct circumscribed and inscribed figures made up of element-prisms,

such that

(circumscr. fig.) — (inscr. fig.) <X.

Therefore (inscr. fig.) > (circumscr. fig. —X),

and a *fortiori >* (portion of cyl. — X).

It follows that

(prism cut off) <3.(inscribed figure).

Considering now the element-prisms in the prism cut off and those in the

inscribed figure respectively, we have

el. prism (N) : el. prism (0) =.11 : mo2

= .1/N : *L* [as in Prop. 13]

=el. rect.. (N) : el. rect. *(L).*

It follows that

I{ el. prism (N)} : { el. prism (0)} =Ile!. rect. *(N)) : ≥* {el. rect. *(L)}.*

(There are really two more prisms and rectangles in the first and third than

there are in the second and fourth terms respectively; but this makes no differ-

ence because the first and third terms may 1)e multiplied by a common factor

as n/(n-2) without affecting the truth of the proportion. Cf. the proposition

from On *Conoids and Spheroids* quoted on p. **571** above.)

Therefore

(prism cut off) : (figure inscr. in portion of cyl.)

= (rect. *GD) :* (fig. inscr. in parabola).

But it was proved above that

(prism cut off) <3(fig. inscr. in portion of cyl.);

therefore (rect. *GD)* <(fig. inscr. in parabola),

and, a *fortiori* (rect. *GD)<(parabolic* segmt.):

which is impossible, since

(rect. *GD) =* i(parabolie segmt.).

Therefore (portion of cyI.) is *not* greater than 3(prism cut off).

1. In the second lacuna must have come the beginning of the next *reductio ad a.bsurdum* demolishing the other possible assumption that the portion of the cylinder is < 3 of the prism cut off.

In this case our assumption is that

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(prism cut off) > (portion of cylinder); and we circumscribe and inscribe figures made up of element-prisms, such that

(prism cut off) > circumscr. about portion of cyl.).

We now consider the element-prisms in the prism cut off and in the circum-

scribed figure respectively, and the same argument as above gives

(prism cut off) : (fig. circumscr. about portion of cyl.)

= (rect. *GD) :* (fig. circumscr. about parabola),

whence it follows that

(rect. *GD)>4(fig.* circumscribed about parabola),

and, a *fortiori,*

(rect. *GD) >* 4(parabolic segment):

which is impossible, since

(rect. *GD)* = (parabolic segmt.).

Therefore

(portion of cyl.) is *not* less than ;(prism cut off).

But it was also proved that neither is it greater;

therefore (portion of cyl.) = 3(prism cut off)

=11(original prism).]

[PROPOSITION 15]

[This proposition, which is lost, would be the mechanical investigation of the second of the two special problems mentioned in the preface to the treatise, namely that of the cubature of the figure included between two cylinders, each of which is inscribed in one and the same cube so that its opposite bases are in two opposite faces of the cube and its surface touches the other four faces.

Zeut.hen has shown how the mechanical method can be applied to this case.

In the accompanying figure V IV YX is a section of the cube by a plane (that of the paper) passing through the axis *BD* of one of the cylinders inscribed in the cube and parallel to two opposite faces.

**X**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
| **13** | **0** | **Q** | **SR** |  |
| **/** |  |  | **K** |  |

**V11**

The same plane gives the circle *ABCD as* the section of the other inscribed cylinder with axis per­pendicular to the plane of the paper and extending on each side of the plane to a distance equal to the radius of the circle or half the *M* side of the cube.

A *C* is the diameter of the circle which is perpendicular to *BD.*

Join *AB, AD* and produce them to meet the tangent at *C* to the circle in *E, F.*

Then *EC =CF =CA.*

Let *LG* be the tangent at *A,* and complete the rectangle *EFGL.*

Draw straight lines from A to the four corners of the section in which the plane through *BD* perpendicular to AK cuts the cube. These straight lines, if

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produced, will meet the plane of the face of the cube opposite to ***A*** in four points forming the four corners of a square in that plane with sides equal to ***EF*** or double of the side of the cube, and we thus have a pyramid with A for vertex and the latter square for base.

Complete the prism (parallelepiped) with the same base and height as the pyramid.

Draw in the parallelogram ***LF*** any straight line ***MN*** parallel to *EF,* and through ***MN*** draw a plane at right angles to ***AC.***

This plane cuts—

1. the solid included by the two cylinders in a square with side equal to *OP,*
2. the prism in a square with side equal to ***MN,*** and
3. the pyramid in a square with side equal to ***QR.***

Produce ***CA*** to ***H,*** making ***HA*** equal to ***AC,*** and imagine *HC* to be the bar

of a balance.

Now, as in Prop. 2, since ***MS= AC, QS= AS,***

***MS •SQ=CA •AS***

***=A02***

***= 0S2-1-SQ2.***

Also ***HA : AS = CA : AS***

***=MS : SQ***

***= MS2 : MS •SQ***

***= MS2 :*** (082+8Q2), from above,

= 3/N2 : *(0P2+QJ?2)*

= (square, side ***MN) :*** (sq., side OP+sq., side ***QR).***

Therefore the square with side equal to ***MN,*** in the place where it is, is in

equilibrium about ***A*** with the squares with sides equal to ***OP, QR*** respectively

placed with their centres of gravity at ***H.***

Proceeding in the same way with the square sections produced by other

planes perpendicular to ***AC,*** we finally prove that the prism, in the place where

it is, is in equilibrium about A with the solid included by the two cylinders and

the pyramid, both placed with their centres of gravity at ***H.***

Now the centre of gravity of the prism is at ***K.***

Therefore ***HA : AK =*** (prism) : (solid+ pyramid)

or 2 : I = (prism) : (solid-Hi prism).

Therefore 2 (solid)+1(prism). (prism).

It follows that

(solid included by cylinders) = 1r(prism)

= 3 (cube). Q.E.D.  
There is no doubt that Archimedes proceeded to, and completed, the rigor­ous geometrical proof by the method of exhaustion.

As observed by Prof. C. Juel (Zeuthen l.c.), the solid in the present proposi­tion is made up of 8 pieces of cylinders of the type of that treated in the pre­ceding proposition. As however the two propositions are separately stated, there is no doubt that Archimedes' proofs of them were distinct.

In this case ***AC*** would be divided into a very large number of equal parts and planes would be drawn through the points of division perpendicular to ***AC.*** These planes cut the solid, and also the cube V )7, in square sections. Thus we can inscribe and circumscribe to the solid the requisite solid figures made up of element-prisms and differing by less than any assigned solid magnitude; the

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prisms have square bases and their heights are the small segments of AC. The element-prism in the inscribed and circumscribed figures which has the square equal **to *OP2*** for base corresponds to an element-prism in the cube which has for base a square with side equal to that of the cube; and as the ratio of the element-prisms is the ratio *0S2* : *BK2,* we can use the same auxiliary parabola, and work out the proof in exactly the same way, as in Prop. 14.]